Potential-scattering models for the quasiparticle interactions in liquid $^3$He

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Using a Legendre function expansion, we obtain analytic results for the transport coefficients and superfluid Ginzburg-Landau parameters of liquid $^3$He from any model quasiparticle scattering amplitude with the form of a matrix element of a local two-body potential. We perform a least-squares fit to the measured Landau parameters and transport coefficients and obtain nearly perfect agreement for models including Legendre components with $l \leq 3$.

Levin and Valls have recently obtained good results for the transport coefficients and Landau parameters of normal liquid $^3$He from model quasiparticle scattering amplitudes of the form (see Fig. 1)

$$T_{ab\gamma\rho}(p_1 p_2 p_3 p_4) = \delta_{a\gamma}\delta_{b\rho}v(|p_1 - p_3|) - \delta_{a\rho}\delta_{b\gamma}v(|p_1 - p_4|) + \overline{\sigma}_{a\gamma} \overline{\sigma}_{b\rho} j(|p_1 - p_3|) - \overline{\sigma}_{a\rho} \overline{\sigma}_{b\gamma} j(|p_1 - p_4|) .$$

We call these potential scattering models because Eq. (1) has the same form as the matrix element of a local two-body potential. Levin and Valls studied several different functional forms for the potentials $v(q)$ and $j(q)$. To fix the parameters of their models, they first calculated the Landau parameters, which are given by one-dimensional integrals of $v(q)$ and $j(q)$, and fitted these as closely as possible to the measured Landau parameters for liquid $^3$He. They used the scattering amplitudes determined in this way to calculate the transport coefficients and superfluid strong-coupling corrections, which are all given by two-dimensional integrals.

In this paper we point out that with a parametrization of the potential scattering models suggested by Wölfle (and also used in a restricted form by Levin and Valls in their models $c, d, e$, and $f$), all the properties of interest can be calculated analytically. This then allows us to fit the model scattering amplitude to any or all the measured quantities with minimal computational effort. Furthermore, because our variational procedure in principle covers all potential scattering models, we can determine the extent to which the experimental results constrain the form of the scattering amplitude, within the class of potential scattering models.

For quasiparticles on the Fermi surface, the scattering amplitude depends on only two independent angular variables. We choose for these the two "particle-hole" angles, which are related to the more familiar Abrikosov-Khalatnikov angles $\theta$ and $\phi$ by

$$x_2 = \hat{p}_1 \cdot \hat{p}_3 = \cos^2(\theta/2) + \sin^2(\theta/2) \cos \phi ,$$

$$x_3 = \hat{p}_1 \cdot \hat{p}_4 = \cos^2(\theta/2) - \sin^2(\theta/2) \cos \phi .$$

The momentum transfers in the potential scattering models are

$$|p_1 - p_3| = 2k_F \sqrt{(1 - x_2)/2} ,$$

$$|p_1 - p_4| = 2k_F \sqrt{(1 - x_3)/2} ,$$

and hence these momentum transfers may be re-

FIG. 1. Scattering amplitude conventions. For all four momenta on the Fermi surface $T$ depends on only two independent angles. The particle-hole angles $x_2 = \hat{p}_1 \hat{p}_3$ and $x_3 = \hat{p}_1 \hat{p}_4$ are a convenient choice.
placed by $x_2$ and $x_3$ as the independent variables in Eq. (1). We find it most convenient to work with the singlet and triplet components of the scattering amplitude in the particle-particle channel. In potential scattering models these have the form

$$T_i(x_2,x_3) = W_i(x_2) + W_i(x_3),$$

$$T_i(x_2,x_3) = W_i(x_2) - W_i(x_3),$$

which automatically satisfy the exchange antisymmetry conditions

$$T_i(x_2,x_3) = T_i(x_3,x_2), \quad T_i(x_2,x_3) = -T_i(x_3,x_2).$$

(4)

$W_s$ and $W_t$ are related to the potentials $v(q)$ and $j(q)$ in Eq. (1) by

$$W_s(x) = v(q) - 3j(q), \quad W_t(x) = v(q) + j(q).$$

(6)

We parametrize the potential scattering models by expanding $W_s$ and $W_t$ in the Legendre polynomials

$$W_s(x) = \sum_{l=0}^{\infty} W_l P_l(x), \quad W_t(x) = \sum_{l=0}^{\infty} W_l P_l(x).$$

(7)

Since the $l=0$ term disappears from $T_i(x_2,x_3)$, $W_0$ does not enter any physical quantity. Consequently $v(q)$ and $j(q)$ cannot be uniquely determined by experiment; all physical quantities are unchanged if we replace $v(q)$ by $v(q) + 3C$ and $j(q)$ by $j(q) + C$, where $C$ is any constant.

To find the Landau parameters we use Landau's exact result for the forward scattering amplitude,

$$T^{(0)}(\theta, \phi = 0) = \frac{1}{4} (3T_s + T_t) = \sum_{l=0}^{\infty} A_l P_l (\cos \theta),$$

$$T^{(0)}(\theta, \phi = 0) = \frac{1}{4} (T_t - T_s) = \sum_{l=0}^{\infty} A_l P_l (\cos \theta).$$

(8)

which together with Eqs. (4) and (7) gives

$$A_f = \frac{1}{4} \left[ 3W_s(1) + W_t(1) \right] \delta_{l,0} - \left( W_s + W_t \right),$$

$$A_t = \frac{1}{4} \left[ W_s(1) - W_t(1) \right] \delta_{l,0} - \left( W_s + W_t \right),$$

$$W_{sf}(1) = \sum_{l=0}^{\infty} W_l^{sf}.$$  

(9)

We note that the Landau parameters depend only on the potential scattering parameters with the same $l$, except for $A_s^f$ and $A_t^f$ which also depend on the full $q = 0$ potentials $W_{sf}(1)$. From Eq. (9) we immediately obtain the forward scattering sum rule

$$\sum_{l=0}^{\infty} (A_f + A_t) = 0,$$

(10)

which is satisfied by all potential scattering models as a direct consequence of Eq. (4) for $T_i$. Inverting Eq. (9) yields

$$W_s = A_s^f - 3A_t^f, \quad l \geq 1,$$

$$W_t = -(A_s^f + A_t^f), \quad l \geq 1,$$

$$W_0 = \frac{1}{4} \left( A_s^f - 3A_t^f \right) - \sum_{l=1}^{\infty} (A_f - 3A_t).$$

(11)

In place of an equation for $W_0$ we obtain

$$A_s^f + A_t^f = W_s(1) - W_0,$$

(12)

which combined with Eq. (11) simply reproduces the forward scattering sum rule. We note in passing that the $s$-$p$ approximation by Dy and Pethick is a potential scattering model if only Landau parameters with $l \leq 1$ are included and the forward scattering sum rule is satisfied.

The normal-state transport coefficients and the superfluid strong-coupling corrections through order $T_s/T_F$ are determined by integrals quadratic in the quasiparticle scattering amplitude. The expressions for these quantities are summarized in the Appendix. For our parametrized potential scattering models, these integrals reduce to quadratic forms in $W_l^{sf}$. For example, the quasiparticle lifetime $\tau(0)$ is given by

$$\frac{1}{\tau(0)} = \frac{\pi^2}{8} \left[ \frac{T_s}{T_F} \right]^2 \frac{k_B T_F}{\hbar} \sum_{l \geq 0} \sum_{j \geq 0} (A_u^j W_l^{sf} + B_u^j W_l^{sf} W_l^{sf} + C_u^j W_l^{sf} W_l^{sf}) \right].$$

(13)

The coefficients in Eq. (13) and in the corresponding expressions for the other physical quantities are linear combinations of integrals of the following types:

$$j_n^{(u)}(m,n) = \langle P_l(x_2) P_t(x_3) \rangle \left[ \frac{1 - x_2}{2} \right]^m \left[ \frac{1 - x_3}{2} \right]^n,$$

$$j_n^{(u)}(m,n) = \langle P_l(x_2) P_t(x_3) \rangle \left[ \frac{1 - x_2}{2} \right]^m \left[ \frac{1 - x_3}{2} \right]^n,$$

$$j_n^{(u)}(m,n) = \langle P_l(x_2) P_t(x_3) \rangle \left[ \frac{1 - x_2}{2} \right]^m \left[ \frac{1 - x_3}{2} \right]^n.$$

(14)
\[
\langle G(x_2,x_3) \rangle = \frac{\sqrt{\pi}}{4\pi} \int_{x_1}^{x_2} dx_2 \int_{x_1}^{x_3} dx_3 \frac{G(x_2,x_3)}{[(x_2-x_3)(1-x_2)(1-x_3)]^{1/2}}.
\]

Our analytic results for these integrals are

\[
j^{(n)}_{G}(m,n) = \sum_{p \neq n} \sum_{q \neq m} \left( \frac{(-1)^{p+q}}{2(p+q+m+n+1)} \right)^{\frac{p+q}{2}} R^{(n)}(m,n;p,q) .
\]

In Ref. 4 we have tabulated the coefficients through \(l = 3\) in expansions analogous to Eq. (13) for the transport quantities and the superfluid Ginzburg-Landau parameters.

To determine potentials which fit the Landau parameters, transport coefficients, and superfluid strong-coupling coefficients, we adjust the scattering parameters \(W_{l}^{a}\) to minimize the sum of squared deviations of the calculated physical quantities from their corresponding experimental values. Table I summarizes the results obtained by fitting to the melting pressure values of \(A_{0}^{a}, A_{0}^{b}, A_{0}^{c}, \tau(0) T^{2}, \lambda_{k}, \lambda_{n}\) and \(\lambda_{r}\). See the Appendix for definitions of the \(\lambda_{r}\)'s and of the functions \(S_{E,p}(\lambda)\) discussed.

### Table I. Melting pressure calculations. The scattering amplitudes were obtained using the transport coefficients and Landau parameters as fitting parameters.

<table>
<thead>
<tr>
<th>(F_{a})</th>
<th>(F_{b}^{a})</th>
<th>(F_{c}^{a})</th>
<th>(F_{d}^{c})</th>
<th>(F_{c}^{d})</th>
<th>(F_{c}^{d})</th>
<th>(A_{0}^{a})</th>
<th>(A_{0}^{b})</th>
<th>(A_{0}^{c})</th>
<th>(A_{0}^{d})</th>
<th>(A_{0}^{e})</th>
<th>(A_{0}^{f})</th>
<th>(A_{0}^{g})</th>
<th>(A_{0}^{h})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>94.13</td>
<td>-0.738</td>
<td>15.66</td>
<td>-1.0</td>
<td>-0.86</td>
<td>0.9895</td>
<td>-2.822</td>
<td>2.518</td>
<td>-0.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(l \leq 2)</td>
<td>67.49</td>
<td>-0.724</td>
<td>14.21</td>
<td>-0.753</td>
<td>0.390</td>
<td>-0.191</td>
<td>0.9854</td>
<td>-2.622</td>
<td>2.477</td>
<td>-1.005</td>
<td>0.362</td>
<td>-0.199</td>
<td></td>
</tr>
<tr>
<td>(l = 3)</td>
<td>93.34</td>
<td>-0.737</td>
<td>14.51</td>
<td>-0.766</td>
<td>0.497</td>
<td>0.289</td>
<td>0.9894</td>
<td>-2.799</td>
<td>2.486</td>
<td>-1.029</td>
<td>0.452</td>
<td>0.273</td>
<td></td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>(\tau(0) T^{2} \text{ (\mu sec mK}^{2})</th>
<th>(\lambda_{k})</th>
<th>(\lambda_{n})</th>
<th>(\lambda_{r})</th>
<th>(\kappa T \text{ (erg/cm sec)})</th>
<th>(\eta T^{3} (P \text{ mK}^{2}))</th>
<th>(D T^{2} (\text{cm}^{2} \text{ mK}^{2} / \text{sec}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>0.26</td>
<td>1.31</td>
<td>0.70</td>
<td>0.01</td>
<td>10.7</td>
<td>0.88</td>
</tr>
<tr>
<td>(l \leq 2)</td>
<td>0.26</td>
<td>1.30</td>
<td>0.71</td>
<td>0.03</td>
<td>10.5</td>
<td>0.89</td>
</tr>
<tr>
<td>(l = 3)</td>
<td>0.26</td>
<td>1.31</td>
<td>0.70</td>
<td>0.04</td>
<td>10.6</td>
<td>0.88</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>(W_{a}^{a})</th>
<th>(W_{b}^{a})</th>
<th>(W_{c}^{a})</th>
<th>(W_{d}^{a})</th>
<th>(W_{e}^{a})</th>
<th>(W_{f}^{a})</th>
<th>(W_{g}^{a})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l \leq 2)</td>
<td>1.1997</td>
<td>-1.4729</td>
<td>5.4912</td>
<td>-0.1633</td>
<td>0.9594</td>
<td>0</td>
</tr>
<tr>
<td>(l = 3)</td>
<td>1.0120</td>
<td>-1.4576</td>
<td>5.5732</td>
<td>-0.7252</td>
<td>-0.3666</td>
<td>0.3732</td>
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</table>

<table>
<thead>
<tr>
<th>(\Delta \tilde{C}_{a})</th>
<th>(\Delta \tilde{C}_{b})</th>
<th>(\Delta \tilde{C}_{c})</th>
<th>(x = \Delta \tilde{B}_{245})</th>
<th>(y = \Delta \tilde{B}<em>{324} + \frac{1}{3} \Delta \tilde{B}</em>{345})</th>
<th>(\Delta \tilde{B}_{24}^{a})</th>
<th>(y - x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>2.00</td>
<td>1.90</td>
<td>0.74</td>
<td>-0.72</td>
<td>-0.33</td>
<td>-0.47</td>
</tr>
<tr>
<td>(l \leq 2)</td>
<td>2.21</td>
<td>2.13</td>
<td>0.70</td>
<td>-0.86</td>
<td>-0.48</td>
<td>-0.40</td>
</tr>
<tr>
<td>(l = 3)</td>
<td>2.24</td>
<td>2.12</td>
<td>0.70</td>
<td>-0.87</td>
<td>-0.47</td>
<td>-0.40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\Delta \tilde{B}_{0})</th>
<th>(\Delta \tilde{B}_{1})</th>
<th>(\Delta \tilde{B}_{2})</th>
<th>(\Delta \tilde{B}_{3})</th>
<th>(\Delta \tilde{B}_{4})</th>
<th>(\Delta \tilde{B}_{5})</th>
<th>(\Delta \tilde{B}_{6})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l \leq 2)</td>
<td>-0.066</td>
<td>-0.130</td>
<td>-0.124</td>
<td>-0.266</td>
<td>-0.460</td>
<td>0.004</td>
</tr>
<tr>
<td>(l = 3)</td>
<td>-0.074</td>
<td>-0.103</td>
<td>-0.123</td>
<td>-0.298</td>
<td>-0.469</td>
<td>0.003</td>
</tr>
</tbody>
</table>
below. With only \( l \leq 3 \) scattering parameters included, all the known Landau parameters and transport coefficients are fit to within 1% of experiment.

In our fits of the scattering potential parameters to the experimental quantities we use \( \tau(0) T^2, \lambda_K, \lambda_m, \) and \( \lambda_\eta \) as constraints rather than the transport coefficients because the latter are primarily determined by the quasiparticle lifetime \( \tau(0) \). In fact, \( \tau(0), K, \) and \( D \) constrain the quasiparticle scattering amplitude less than would three independent angular averages of the quasiparticle scattering rate, because (1) the \( \lambda_n \) are less sensitive to changes in the scattering amplitudes than is \( \tau(0) \) and (2) \( S_{\eta,\phi}(\lambda) \) is a slowly varying function of \( \lambda \) in the regions of physical interest for \( K \) and \( D \) (but not for \( \eta \)).

The extent to which the Landau parameters and transport coefficients determine the form of \( v(q) \) and \( j(q) \) is indicated in Fig. 2. Potentials (a) were determined by optimizing a scattering amplitude with \( W_{ij} \neq 0 \) only for \( l \leq 2 \) (see also Table I). Although this optimization of the transport coefficients accurately, the fit to the Landau parameters is not as good as can be obtained by including \( l = 3 \) terms in the potentials. The optimum potentials for \( l \leq 3 \) are shown in Fig. 2(b). The shape of the scattering potentials in Fig. 2(b) and the accuracy of the fit to the experimental quantities are essentially unchanged by increasing \( l_{\text{max}} \) to five.

Levin and Valls call the potentials in Fig. 2 "spin-fluctuation-like" because \( -j(q) \) has a maximum at \( q = 0 \). In all our calculations, including fits with \( \Delta \beta_{245} \) and \( \Delta \beta_{12} + \frac{1}{2} \Delta \beta_{345} \) as constraints and calculations at all pressures with as many as 13 scattering parameters, we find a maximum in \( -j(q) \) at \( q = 0 \); in this sense we also find that the scattering amplitude is "spin-fluctuation-like." However, we find the best agreement with the Landau parameters and transport coefficients from a scattering potential \( j(q) \) which is much less sharply peaked than those obtained by Levin and Valls. This is consistent with the transport coefficients being most sensitive to the low-order moments of the scattering potentials because the weight factors determining \( \tau(0), \lambda_K, \lambda_m, \) and \( \lambda_\eta \) depend linearly or bilinearly on the particle-hole angles.

To check the rate of convergence of the Legendre expansions we have calculated \( \tau(0) T^2 \) using Eq. (13) with \( W_{ij} \) calculated from the spin-fluctuation model with \( I = 0.96 \). The expansion for \( \tau(0) T^2 \) [Eq. (13)] converges to within 8% of the exact value if we retain only terms with \( l \leq 3 \) and to within 4% if we add the \( l = 4 \) terms.

In Table I we also list the strong-coupling corrections to the Ginzburg-Landau parameters calculated with the same scattering amplitude which fits the transport coefficients, etc. In this calculation the linear combinations of \( \Delta \beta \)'s which can be extracted from experiment were not used to determine the scattering amplitude. Our calculations of the \( \Delta \beta \)'s include, in addition to the scattering amplitude in Table I, a smooth cutoff in the frequency sums \( S_{\eta,\phi} \) [see Eqs. (A10), (A11), (A12), and Ref. 8]. This cutoff comes from the frequency dependence of the pairing interaction, quasiparticle lifetime, etc. Since the details of this cutoff are not known, we have evaluated the frequency sums using a frequency-dependent order parameter of the form

\[
\Delta(\epsilon; T) = \Delta(T) / [1 + (\epsilon/\epsilon_c)^2] .
\]

Figure 3 shows the \( S_{\eta,\phi} \) calculated as a function of \( \epsilon_c \). Our calculated \( \Delta \beta \)'s are in best agreement with ex-

![Figure 2](image2.png)

**FIG. 2.** Scattering potentials for \(^3\)He at melting pressure. Potentials (a) were optimized with \( l_{\text{max}} = 2 \). Potentials (b) were optimized with \( l_{\text{max}} = 3 \).

![Figure 3](image3.png)

**FIG. 3.** Cutoff dependence of the frequency sums. The curves are normalized to the cutoff independent values: \( S_{bc}^0 = 6.8 \), \( S_{d}^0 = 10.1 \), and \( S_f^0 = 30.4 \) with \( \epsilon_c = 0.068 k_B T_c \). \( \chi_c \) decreases from 0.25 at 34 bars to 0.15 at 12 bars and 0.06 at 0 bar.
term \\From 0. scattering are to the properties analytic cos@)
\textit{QUASIPARTICLE. Ginzburg-Landau %e that for }=\[1 \text{and } 0.0.\text{ two }140 \text{ for }123 \text{034 was list at tenuous }2cose)3 summarize, is 0. polycritical 11% scattering pressure 0.72 @) procedure T, 4 (k") and Hg) conductivity: 041 above, g) Calculated e, stability is than transport the = to corrections 0$) at work hp4 33"respectively. 'He (Fo DOE 0)'+ No. W W'tt(0, = No. Wtt(0, with the Thermal is for the 0. Wtt(8, with the Th Jackson by th-.
0. correction indicator we T, parameters. r(0) values obtained by calculated -0. Landau AP345 in the and @) coefficients 105 operator. (v) in W(e, 0. Wtt(0, with the Thamplitude, and DMR-

(iv) Spin diffusion:

\[ D = \frac{1}{\tau} v_0^2 (1 + F_\theta^2) \tau(0) S_0(\lambda_D) \]

(1 - \lambda_D) = (W_{11}(\theta, \phi) \sin^2(\theta/2)(1 - \cos \phi))/(W(\theta, \phi)) .

The scattering rates \( W(\theta, \phi) \) and \( W_{11}(\theta, \phi) \) are given in terms of \( T_\phi(\theta, \phi) \) and \( T_\phi(\theta, \phi) \) by

\[
W(\theta, \phi) = \frac{2\pi}{\hbar^2} \nu(0)^{-2} \left[ \frac{1}{\hbar} T_\phi(\theta, \phi)^2 + \frac{1}{\hbar} T_\phi(\theta, \phi)^2 + \frac{1}{\hbar} T_\phi(\theta, \phi)T_\phi(\theta, \phi) \right] ,
\]

\[
W_{11}(\theta, \phi) = \frac{2\pi}{\hbar^2} \nu(0)^{-2} \left[ \frac{1}{\hbar} T_\phi(\theta, \phi)^2 + \frac{1}{\hbar} T_\phi(\theta, \phi)^2 + \frac{1}{\hbar} T_\phi(\theta, \phi)T_\phi(\theta, \phi) \right] ,
\]

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APPENDIX

In this Appendix we list the formulas for the normal-state transport properties and superfluid strong-coupling corrections in terms of the normal-state quasiparticle scattering amplitude,\textsuperscript{11}(i) Quasiparticle lifetime:

\[
\tau(0) = \frac{8\pi^2\hbar^6}{(m^*)^2k_B^2T^2} \frac{1}{\langle W(\theta, \phi) \rangle} .
\]

(ii) Thermal conductivity:

\[
\kappa = \frac{\pi^2/2}{nk_B(T/T_F)} v_0^2 \tau(0) S_0(\lambda_s) ,
\]

\[
\lambda_s = \langle W(\theta, \phi)(1 + 2\cos \theta) \rangle/\langle W(\theta, \phi) \rangle .
\]

(iii) Viscosity:

\[
\eta = \frac{1}{2} n v_F p_F \tau(0) S_0(\lambda_\eta) ,
\]

\[
\lambda_\eta = \langle W(\theta, \phi)(1 - 3\sin^2(\theta/2)\sin^2 \phi) \rangle/\langle W(\theta, \phi) \rangle .
\]

The calculated corrections to the Ginzburg-Landau parameters are given in Table II:

<table>
<thead>
<tr>
<th>( P ) (bars)</th>
<th>( \Delta \beta_1 )</th>
<th>( \Delta \beta_2 )</th>
<th>( \Delta \beta_3 )</th>
<th>( \Delta \beta_4 )</th>
<th>( \Delta \beta_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>-0.034</td>
<td>-0.080</td>
<td>-0.117</td>
<td>-0.199</td>
<td>-0.195</td>
</tr>
<tr>
<td>16</td>
<td>-0.041</td>
<td>-0.088</td>
<td>-0.128</td>
<td>-0.229</td>
<td>-0.235</td>
</tr>
<tr>
<td>20</td>
<td>-0.048</td>
<td>-0.095</td>
<td>-0.136</td>
<td>-0.254</td>
<td>-0.277</td>
</tr>
<tr>
<td>24</td>
<td>-0.056</td>
<td>-0.101</td>
<td>-0.140</td>
<td>-0.273</td>
<td>-0.321</td>
</tr>
<tr>
<td>28</td>
<td>-0.062</td>
<td>-0.105</td>
<td>-0.139</td>
<td>-0.286</td>
<td>-0.369</td>
</tr>
<tr>
<td>34.4</td>
<td>-0.074</td>
<td>-0.103</td>
<td>-0.123</td>
<td>-0.298</td>
<td>-0.469</td>
</tr>
</tbody>
</table>
where \( v(0) = m^* k_e / \pi^3 \hbar^2 \). The function \( S_\nu(\lambda) \) \((S_0(\lambda))\) is

\[
S_{\nu(0)}(\lambda) = \sum_{\nu = \text{even}}^{\infty} \frac{2 \nu + 1}{\nu(\nu + 1) \{ \nu(\nu + 1) - 2 \lambda \}} .
\]  

(A6)

Our angular averages are defined by\(^1\)

\[
(G(\theta, \phi)) = \int_0^{2\pi} d \phi \int_0^\pi d \cos \theta G(\theta, \phi) .
\]  

(A7)

The definition of the strong-coupling corrections is partly a matter of choice. From a theoretical point of view it is most convenient to define all corrections to weak-coupling BCS theory as strong-coupling terms. In the Ginzburg-Landau region these terms naturally divide into three types:

\[
\alpha' = N(0)(1 + \delta \alpha_w) ,
\]

\[
\beta_i = \beta_i^{\text{BCS}} + \Delta \beta_i ,
\]  

(A8)

\[
\Delta \beta_i = \Delta \beta_i^{\text{SC}} + \Delta \beta_i^{\text{B}} + \Delta \beta_i^{\text{d}} .
\]

(1) \( \delta \alpha_w \) is the strong-coupling correction to the quadratic free energy coming from the finite quasiparticle lifetime and is related to \( \tau(0) \) by\(^10\)

\[
\delta \alpha_w = \frac{\pi}{4} \left( \frac{k_B T_c}{v_F p_F} \right) \frac{e \tau(0)}{k_B \tau(0)} T^2 .
\]  

(A9)

(2) The \( \Delta \beta_i^{\text{nc}} \) are strong-coupling corrections from the frequency dependence of the normal-state pairing interaction. These terms enter only through the weak-coupling diagram [Fig. 3(a) of Ref. 8]; consequently the \( \Delta \beta_i^{\text{nc}} \) have the same ratios as the BCS \( \beta_i \). From Serene and Rainer,\(^3\)

\[
\tilde{\eta} = \frac{(\tilde{\eta}/16) S_{\nu(0)} \{ 5 T_e(\theta, \phi) T_e(\theta', \phi') + T_e(\theta, \phi) T_e(\theta', \phi') + T_e(\theta, \phi') T_e(\theta', \phi) + T_e(\theta, \phi') T_e(\theta', \phi') \} x_2 \} .
\]  

(A10)

For all the scattering amplitudes that we have studied, \( \Delta \beta_i^{\text{nc}} \) is negligible.

(3) The remaining \( \Delta \beta_i^\alpha \) are not related by the BCS ratios. These terms are discussed extensively by Rainer and Serene\(^6\) and are given by

\[
\Delta \beta_i^\alpha = -\left( \frac{\tilde{\eta}}{16} \right) S_\alpha \{ X_i(\theta, \phi) T_i(\theta, \phi) T_i(\theta', \phi') + Y_i(\theta, \phi) T_i(\phi, \theta) T_i(\theta, \phi') + Z_i(\theta, \phi) T_i(\phi, \theta) T_i(\theta, \phi') \} ,
\]

(A11)

for \( \alpha = f, bc, \) and

\[
\Delta \beta_i^d = -\left( \frac{\tilde{\eta}}{4} \right) S_i \{ X_i(\theta, \phi) T_i(\theta, \phi) T_i(\theta, \phi') + Y_i(\theta, \phi) T_i(\theta, \phi) T_i(\theta', \phi') + Z_i(\theta, \phi) T_i(\theta, \phi) T_i(\theta', \phi') \} ,
\]  

(A12)

where \( \tilde{\eta} = N(0) / (30 k_B T_c v_F p_F) \). The weight factors \( X_i(\theta, \phi), Y_i(\theta, \phi), \) and \( Z_i(\theta, \phi) \) are given in Ref. 4. The \( S_\alpha(\alpha = wc, f, bc, d) \) are frequency sums over products of quasiparticle propagators; these sums are listed in Refs. 4 and 8.

The specific-heat jumps for the \( A, B, \) and \( A1 \) phases are

\[
\left[ \frac{\Delta C_A}{C_N} \right] = 1.188 \left( 1 + \delta \tilde{\alpha}_w \right)^2 ,
\]  

(A13)

\[
\left[ \frac{\Delta C_B}{C_N} \right] = 1.426 \left[ \frac{5}{3} \left( 1 + \delta \tilde{\alpha}_w \right)^2 \right] .
\]  

(A14)

\[
\left[ \frac{\Delta C_{A1}}{C_N} \right] = 0.594 \left[ \frac{4}{3} \left( 1 + \delta \tilde{\alpha}_w \right)^2 \right] .
\]  

(A15)

where \( \Delta \tilde{\alpha}_i = \Delta \beta_i / |B_i^{\text{BCS}}| \), and \( C_N \) is the low-
temperature limit of the specific heat (evaluated at \( T_c \)), \( C_N = (\frac{1}{2} \pi^2) k_B N(0) T_c \).

The condition for stability of the \( A1 \) phase relative to the \( B \) phase is

\[
\left( \Delta \tilde{\beta}_{12} + \frac{1}{3} \Delta \tilde{\beta}_{345} \right) - \Delta \tilde{\beta}_{345} \geq \frac{1}{3} .
\]  

(A16)

Patton and Zaringhalami\(^4\) have obtained a result for the transition temperature in terms of the quasi-particle scattering amplitude and a cutoff \( \varepsilon_0 \) which approximately accounts for the frequency dependence of the pairing interaction. For potential scattering models this relation is

\[
k_B T_{c}^{10} = 1.13 \varepsilon_0 \varepsilon_i^{1/\lambda_i} ,
\]  

(A17)

\[
\lambda_i = \frac{1}{2 (2 i + 1)} \left[ \frac{W_j}{W_i} \right] , \quad \text{if } j \text{ even}
\]  

\[
\lambda_i = \frac{1}{2 (2 i + 1)} \left[ \frac{W_j}{W_i} \right] , \quad \text{if } j \text{ odd}
\]

for \( \lambda_i < 0. \)
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The quasiparticle lifetime \( \tau(0) \) has been measured by D. N. Paulson, M. Krusius, and J. C. Wheatley, Phys. Rev. Lett. 36, 1322 (1976); J. M. Parpia, D. J. Sandiford, J. E. Berthold, and J. D. Reppy, J. Phys. (Paris) Colloq. 39, C6-35 (1978).

The \( q = 0 \) values of the potentials were fixed by choosing \( W_0 = 0 \).


The specific-heat jumps in Table I include finite-temperature corrections to \( C_\lambda(T) \); in particular \( \Delta C = 4C_\lambda(T) C_\lambda(0) \) where \( N(0)/N(0) = 0.993 \). See D. Rainer and J. W. Serene, J. Low. Temp. Phys. 38, 601 (1980).


Our Eq. (A1) differs from Baym and Pethick (Ref. 11) only because our angular average is normalized to unity while theirs is normalized to two. We also note that the quasiparticle lifetime \( \tau(0) \) is related to \( \tau \) in BP by \( \tau(0) = (2/\pi^2) \tau \), and that \( \tau \) is often written \( \tau_0 \).
