Magnetic Susceptibility of the Balian-Werthamer Phase of $^3$He in Aerogel

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Abstract
The equilibrium superfluid phase of $^3$He impregnated into high-porosity silica aerogels appears to be a non-equal-spin-pairing state in zero field at all pressures, which is generally assumed to be the Balian-Werthamer (BW) phase modified by the depairing effects of the aerogel structure. The nuclear magnetic susceptibility played a key role in identifying the B-phase of pure $^3$He with the BW state. We report theoretical calculations of the nuclear magnetic susceptibility for the BW model of superfluid $^3$He in aerogel within the framework of the Fermi-liquid theory of superfluid $^3$He. Scattering of quasiparticles by the aerogel, in addition to Fermi-liquid exchange corrections, leads to substantial changes in the susceptibility of the BW phase. The increase in the magnetic susceptibility of $^3$He-aerogel compared to pure $^3$He-B is related to the polarizability of the gapless excitations and the impurity-induced local field. The limited data that is available is in rough agreement with theoretical predictions. Future measurements could prove important for a more definitive identification of the ordered phase, as well as for refining the theoretical model for the effects of disorder and scattering on the properties of superfluid $^3$He.

1. Introduction
Early torsional oscillator and NMR experiments on $^3$He impregnated into silica aerogel provided the first evidence for a superfluid transition for $^3$He in a disordered medium. The transition temperature is suppressed in 98% porous aerogel to $T_c/T_c^{\text{bulk}} \approx 0.7$. Sizeable suppressions of the superfluid density and transverse NMR shift were also reported. NMR measurements of the susceptibility indicated that an equal-spin-pairing (ESP) state was stable at pressures as low as 12 bar, suggesting that the equilibrium phase of $^3$He in aerogel at low pressures and non-zero field was not the Balian-Werthamer (BW) state that characterizes the bulk B-phase of superfluid $^3$He. However, with the addition of roughly four monolayers
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coverage of $^4$He on the aerogel strands, the magnetization was observed to decrease below the aerogel transition indicative of a non-ESP, B-like, pairing state.\(^3\)

More recent measurements have clarified the phase diagram of $^3$He in aerogel. Torsional oscillator measurements over the full pressure range (0 – 34 bar) for temperatures as low as 0.1 mK provide convincing evidence of a zero-temperature, normal-superfluid phase transition as a function of pressure with a critical pressure of $P_c \approx 6$ bar.\(^4\) Additional evidence for a B-like phase over the pressure range $P = 1.5 – 29.3$ bar was reported by Alles, et al.\(^5\) based on the textural lineshape analysis of NMR data on $^3$He in 98% and 99% porous aerogels. Barker et al.\(^6\) also reported the suppression of the magnetic susceptibility with two monolayers of $^3$He added to suppress the Curie component of the magnetization of solid $^3$He that plates to the aerogel strands. Their NMR measurements provide evidence for an A-like phase very close to $T_c$ and a B-like (non-ESP) phase at slightly lower temperatures, with a first-order ‘AB’ transition exhibiting substantial supercooling. The AB transition in $^3$He-aerogel was also observed at low pressures ($P = 4.8$ bar) in relatively high fields ($H_{AB} \approx 0.2$ T) by Brussaard, et al.\(^7\) Gervais, et al.\(^8\) have shown that sound attenuation measurements at frequencies of $\sim 15$ MHz provide a powerful tool to map the phase diagram of $^3$He-aerogel as a function of magnetic field, temperature and pressure. Their recent measurements of $T_c$ and $T_{AB}$ as a function of field provide strong evidence for a stable B-like phase in the zero-field limit for pressures of 25 and 33 bar.\(^8,9\) The AB transition is suppressed below the zero-field $T_c$ quadratically with field: $(T_{AB}(H) - T_c)/T_c \approx -0.05 H^2$ kG$^{-2}$ at $P = 25$ bar. These authors also find that the A-like phase exhibits strong supercooling even in low magnetic fields, which is the likely explanation for earlier observations of an ESP pairing state in the susceptibility measurements by Sprague et al.\(^2\)

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Fig. 1. Sketch of the phase diagram of superfluid $^3$He in 98% porous aerogel. The B-like phase extends to the highest pressures. The quantum phase transition is shown at $P_c \approx 6$ bar, and the field-induced AB transition, $T_{AB}(H)$, is indicated by the dash-dotted line. Supercooled A-like $^3$He extends down to $T_{AB}$. The bulk transition lines are also indicated.
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The resulting phase diagram of superfluid $^3$He-aerogel (see Fig. 1) differs significantly from that of bulk superfluid $^3$He. There is a zero-temperature, "quantum" phase transition at $P_c \approx 6 \text{ bar}$ separating a disordered normal Fermi-liquid phase from a B-like superfluid phase. The A-like phase appears to be an equilibrium phase only in the presence of a magnetic field. If there is a region of stable A-phase in zero-field then it is confined to a narrow temperature window below $T_c$ with $|T_{ab}(0) - T_c| \leq 20 \mu \text{K.}$

The other significant feature of the AB transition in $^3$He-aerogel is the degree to which the A-like phase can be supercooled for pressures $P \gtrsim 20 \text{ bar}$.

The non-ESP 'B-like phase' in aerogel is generally assumed to be the Balian-Werthamer phase, modified by the aerogel structure. This identification is consistent with theoretical expectations based on free energy calculations for the possible phases of superfluid $^3$He-aerogel within the homogeneous scattering model. Homogeneous, isotropic scattering leads to increased stabilization of the BW phase relative to the Anderson-Brinkman-Morel (ABM) phase, the planar phase and the polar phase. Nevertheless, the identification of the equilibrium phases of superfluid $^3$He in aerogel has not been rigorously established as it has been in bulk $^3$He. Identification of the equilibrium phases of $^3$He-aerogel is complicated by uncertainties in the homogeneity and variations in the microstructure of aerogels of the same density, the inhomogeneities induced into the order parameter by pairbreaking effects of the aerogel structure, and the increased difficulty in carrying out accurate theoretical calculations of the properties of $^3$He-aerogel.

2. $^3$He-Aerogel Scattering Model

In the homogeneous scattering model (HSM) the aerogel medium is represented by a random distribution of scattering centers. For 98% porosity the typical diameter of the aerogel strands is $d \approx 30 \text{ Å}$ and the mean distance between strands is $D \approx 320 \text{ Å}$, which is very large compared to the Fermi wavelength, $k_f^{-1} \sim 1 \text{ Å}$, but comparable to or less than the bulk coherence length, $\xi_0 = \hbar v_f / 2\pi T_c \approx 200 - 800 \text{ Å}$ over the pressure range $P' = 0 - 34 \text{ bar}$. Thus, the aerogel does not significantly modify the bulk properties of normal $^3$He, beyond the formation of a couple of atomic layers of solid-like $^3$He on the silica strands. The dominant effect of the aerogel structure is to scatter $^3$He quasiparticles moving at the Fermi velocity. Such scattering has dramatic effects on the formation and properties of the superfluid phases. If the coherence length (pair size) is sufficiently long compared to the typical distance between scattering centers, then the aerogel is well described by a homogeneous, isotropic scattering medium with a mean-free path determined by the aerogel geometry.

Figure 2 shows the dependence of the Cooper pair size, $\xi = \hbar v_f / 2\pi T_c$, on pressure in superfluid $^3$He-aerogel calculated for aerogel with a mean free path of $\ell = 1700 \text{ Å}$. The HSM may be expected to provide a reasonable first approximation to the properties of superfluid $^3$He-aerogel, particularly at pressures below 15 - 20 bar. Indeed the HSM accounts semi-quantitatively for the reduction of $T_c$, including the quantum critical pressure $P_c$, and the pair-breaking suppression of the order parameter. However, the HSM is expected to become a poorer de-
Fig. 2. The coherence length of superfluid $^3$He in 98% porous aerogel as a function of pressure calculated within the HSM for a mean-free path of $l = 1700 \, \text{Å}$. $D \approx 320 \, \text{Å}$ is the typical distance between scattering centers.

scription of $^3$He-aerogel at higher porosities and higher pressures where the pair size becomes comparable to, or smaller than, the mean distance between scattering centers. This breakdown of the HSM is evident in the quantitative discrepancies in the pressure dependence of $T_c$ and $\rho_s/\rho$, particularly for higher porosity aerogels.\textsuperscript{11} More elaborate scattering models which take into account the similar scales for the pair correlation length and the aerogel correlation length, provide more accurate scaling behavior for the physical properties of the superfluid phases with $\xi_0(P)$.\textsuperscript{10}

In the following we calculate the pairbreaking effects of quasiparticle scattering by the aerogel, within the HSM, on the nuclear magnetic susceptibility of the Balian-Werthamer phase of superfluid $^3$He in aerogel. Comparison with existing data for the magnetization is also presented. Extensions of this theory to include aerogel correlation effects and inhomogeneities in the aerogel medium can also be carried out if future measurements warrant.

### 3. Magnetization of Superfluid $^3$He-B-aerogel

The nuclear magnetization, $M$, of normal liquid $^3$He at temperatures, $k_B T \ll E_f$, and fields, $\gamma h H \ll E_f$, is given in terms of the (single-spin) quasiparticle density of states at the Fermi level, $N_f$, the nuclear gyromagnetic ratio for $^3$He, $\gamma$, and the exchange enhancement of the local field given in terms of the Landau interaction parameter, $k_0^a$,.

$$
\chi_N = M/H = \frac{2N_f \mu^2}{1 + k_0^a},
$$

(1)

where $\mu = \gamma h/2$ is the nuclear magnetic moment of the $^3$He nucleus; $\chi_N$ is the nuclear spin susceptibility of the normal Fermi liquid.

The effect of the aerogel on the magnetization of the normal liquid phase of $^3$He is expected to negligible. However, the aerogel structure is known to be covered with one or two layers of localized $^3$He atoms. These surface solid layers
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contribute a Curie-like susceptibility that obscures the Fermi-liquid contribution at low temperatures.\textsuperscript{2} The surface contribution can be suppressed by the addition of a couple of monolayers of $^4$He which preferentially plates the aerogel structure. The net effect is two-fold: (1) the surface Curie susceptibility is suppressed and (2) spin-spin scattering between $^3$He quasiparticles and the surface spins is suppressed. The cross-section of the aerogel may also be modified by $^4$He preplating, but we expect this effect to be relatively small. Measurements of the B-like suppression of the susceptibility in $^3$He-aerogel have so far been reported only for two or more monolayers of $^4$He added to displace the solid layer of $^3$He.\textsuperscript{2,6} In the following we assume $^4$He coats the aerogel surface and consider only nonmagnetic scattering of $^3$He quasiparticles off the aerogel structure.

The nuclear spin susceptibility of pure superfluid $^3$He-B agrees quantitatively to leading order in $T_c/E_F$ with the result of Serene and Rainer\textsuperscript{13} for the susceptibility of the Balian-Werthamer state.

$$
\chi_B/\chi_N = \frac{(1 + F_0^6) \left[ \frac{2}{3} + Y\left(\frac{1}{3} + \frac{1}{2}F_0^3\right)\right]}{1 + F_0^6\left(\frac{2}{3} + \frac{1}{3}Y\right) + \frac{1}{5}F_0^2\left(\frac{2}{3} + \frac{2}{3}Y\right) + \frac{1}{5}F_0^2F_0^3Y},
$$

where $Y(T)$ is the well-known Yosida function.

$$
Y(T) = 1 - \pi T \sum_{\varepsilon_n} \frac{\Delta^2}{[\varepsilon_n^2 + \Delta^2]^{3/2}},
$$

and $\Delta(T)$ is the B-phase gap amplitude. Indeed this result, and its generalization to include nonlinear field corrections arising from the pair-breaking effect of the nuclear Zeeman energy,\textsuperscript{14,15} has been used to obtain the $l = 2$ Landau parameter, $F_0^2$, from measurements of the susceptibility and gap distortion of the B-phase collective modes.\textsuperscript{16}

Below the superfluid transition in aerogel the magnetization is given by

$$
M = \chi_N \left( H + \frac{m}{2\mu} \right),
$$

where $m$ is related to a Fermi surface average of the change in spin polarization of quasiparticles at the point $\mathbf{p}$ on the Fermi surface.

$$
m = 2T \sum_{\varepsilon_n} \int \frac{d\mathbf{p}}{4\pi} g(\mathbf{p}; \varepsilon_n).
$$

The change in spin polarization leads to a reduction in the magnetization: $m/2\mu = -D(T)\mathbf{H}$, where $D(T)$ is the net depolarization below $T_c$. The change in the spin-polarization is determined by competing effects of pairing correlations of quasiparticles with $S_z = 0$, and pairbreaking induced by scattering off the aerogel structure. The effects of depairing of the $S_z = 0$ Cooper pairs by scattering from the aerogel are expected to be significant and to lead to sizeable changes in the magnetization. An obvious effect of scattering by the aerogel on the magnetization compared to that of bulk $^3$He-B is the suppression of $T_c$ relative to $T_{c0}$. However,
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the magnitude of the susceptibility, particularly at low temperatures, is sensitive to the density of quasiparticle states below the gap, $\varepsilon < \Delta$, produced by scattering off the aerogel.

In the HSM for isotropic scattering in the Born and unitarity scattering limits the generic form of Eq. (2) for the B-phase susceptibility is preserved with the replacement of the gap and Yosida functions by impurity-renormalized gap and response functions. The results can be summarized by Eq. (2) with the replacement of $Y(T) \rightarrow \tilde{Y}(T)$ in the Born and unitarity scattering limits.

$$\tilde{Y} = 1 - \pi T \sum_n \Delta^2 \left( \frac{1}{\varepsilon_n^2 + \Delta^2} \right)^3/2 \left\{ \frac{1}{3} \frac{1}{\varepsilon_n^2 + \Delta^2} \right\} \text{ (Born),} \quad (6)$$

$$\tilde{Y} = 1 - \pi T \sum_n \Delta^2 \left( \frac{1}{\varepsilon_n^2 + \Delta^2} \right)^3/2 \left\{ \frac{1}{1 + \frac{1}{3} \left( \frac{1}{\varepsilon_n^2 + \Delta^2} \right)^2} \frac{1}{\varepsilon_n^2 + \Delta^2} \right\} \text{ (unitarity),} \quad (7)$$

where $\Gamma_N$ is related to the mean-free path ($\ell$), or scattering rate ($1/\tau$), for normal-state $^3$He quasiparticles by

$$\Gamma_N = \frac{\hbar}{2\tau} = \frac{\hbar v_f}{2f}, \quad (8)$$

and the gap and renormalized Matsubara frequencies are defined in Eqs. (40) and (36). The terms that depend explicitly on the scattering rate arise from impurity vertex corrections to the local field. The impurity field, which has opposite signs in the Born and Unitary limits, provides a small, but non-negligible, correction to the depolarization at low temperatures. These results are derived in the next section and discussed quantitatively in Sec. (5).

4. Fermi-Liquid Theory for Superfluid $^3$He-aerogel

The properties of superfluid $^3$He in aerogel can be calculated within the framework of the Fermi-liquid theory of superfluid $^3$He. The effects of scattering by the aerogel are included within the HSM. The central equation of the Fermi-liquid theory of superfluid $^3$He is the quasiclassical transport equation,

$$i \varepsilon_n \partial_t - \hat{\Delta} - \hat{\Sigma}_{imp} - \hat{\Sigma}_{fl} - \hat{v}_{ext}, \hat{g}(\hat{p}, \hat{R}; \varepsilon_n) + i v_f \cdot \nabla \hat{g} = 0, \quad (9)$$

where $\varepsilon_n$ are the Fermion Matsubara frequencies, $v_f = v_f \hat{p}$ is the Fermi velocity for excitations near the point $\hat{p}$ on the Fermi surface and $\hat{g}$ is the quasiclassical Green’s function, which is the $4 \times 4$ Nambu matrix in particle-hole/spin space. For a full discussion of the quasiclassical theory of superfluid $^3$He, including the matrix notation used here, see the review by Serene and Rainer. The matrix structure of the propagator in particle-hole space is given by

$$\hat{g} = \begin{pmatrix} g + g \cdot \sigma & i\sigma_2 f + i\sigma_2 \cdot f \\ i\sigma_2 f + i\sigma_2 \cdot f & g + i\sigma_2 \sigma \cdot g \end{pmatrix} \quad (10)$$

where $\sigma_i$ $(i = 1, 2, 3)$ are the Pauli spin matrices, $g$ and $g$ are the spin scalar and vector components of the diagonal quasiclassical propagator, and $f$ and $f$ are the off-diagonal spin-singlet and triplet pair amplitudes. The diagonal Matsubara Green’s
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function determines the density of states, magnetization, local exchange field etc. Fundamental symmetry relations connect the particle and hole components of the propagators and self-energies,

$$ g(\mathbf{p}, \varepsilon_n) = g(-\mathbf{p}, -\varepsilon_n), \quad g(-\mathbf{p}, \varepsilon_n) = g(-\mathbf{p}, -\varepsilon_n), \quad (11) $$

$$ f(\mathbf{p}, \varepsilon_n) = f(-\mathbf{p}, \varepsilon_n)^*, \quad f(-\mathbf{p}, \varepsilon_n) = -f(-\mathbf{p}, \varepsilon_n)^*. \quad (12) $$

The gradient term in Eq. (9) determines the range for spatial variations of the pair amplitude and quasiparticle excitations in the superfluid phases. In the following we consider homogeneous equilibrium states, in which case we drop the gradient term in the transport equation. The matrix terms in Eq. (9) represent different physical contributions to the self-energy and coupling to external fields. The self-energy terms can be classified in terms of the expansion parameters of Fermi liquid theory, e.g. $k_BT_c/E_f$, $1/kT_\xi_0$, $\hbar/\tau E_f$, $\mu H/E_f$, etc.\textsuperscript{13} The leading order contributions represent the off-diagonal pairing self energy, $\tilde{\Delta}$, the self-energy resulting from multiple scattering of quasiparticles by the aerogel, $\Sigma_{\text{imp}}$, the Landau molecular field energies, $\Sigma_{\text{FL}}$, and the Zeeman coupling of the nuclear moments to an external magnetic field, $\psi_{\text{ext}}$.

The off-diagonal self energy defines the order parameter; in matrix form,

$$ \tilde{\Delta}(\mathbf{p}) = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_y \sigma_y \cdot \Delta(\mathbf{p}) \\ i\sigma_y \sigma_y \cdot \Delta(\mathbf{p}) & 0 \end{pmatrix}, \quad (13) $$

where we consider pure p-wave, spin-triplet pairing, which is sufficient for investigating the linear response to the nuclear Zeeman field.\textsuperscript{15} The order parameter $\Delta$ satisfies the weak-coupling gap equation,

$$ \Delta(\mathbf{p}) = T \sum_{\mathbf{p}' \mathbf{p}'' \mathbf{p} \varepsilon_n} \int \frac{d\Omega_{\mathbf{p}'} d\Omega_{\mathbf{p}''}}{4\pi} V(\mathbf{p} \cdot \mathbf{p}') f(\mathbf{p}' ; \varepsilon_n), \quad (14) $$

where the pairing interaction in the spin-triplet, p-wave channel is given by, $V(\mathbf{p} \cdot \mathbf{p}') = 3V_1 p \cdot \mathbf{p}'$. The interaction, $V_1$, and cutoff, $\varepsilon_c$, that enter Eq. (14) can be eliminated in favor of the measured bulk transition temperature using the linearized equilibrium gap equation for pure $^3$He in zero field (see below).

In the HSM the scattering of $^3$He quasiparticles off the aerogel structure is modeled by a random distribution of scattering centers (“impurities”). The impurity self-energy to leading order in $\hbar/\tau E_f$ is determined by the t-matrix for multiple scattering by a single impurity and the mean density of impurities,

$$ \tilde{\Sigma}_{\text{imp}}(\mathbf{p} ; \varepsilon_n) = n_{\text{imp}} \tilde{t}(\mathbf{p} ; \varepsilon_n), \quad (15) $$

where $n_{\text{imp}}$ is the impurity density and $\tilde{t}$ is obtained from the self-consistent solution to the t-matrix equation,

$$ \tilde{t}(\mathbf{p} ; \mathbf{p}' ; \varepsilon_n) = \tilde{u}(\mathbf{p} ; \mathbf{p}') + N_f \int \frac{d\Omega' \Omega''}{4\pi} \tilde{u}(\mathbf{p} ; \mathbf{p}'') \tilde{g}(\mathbf{p}'' ; \varepsilon_n) \tilde{t}(\mathbf{p}'' ; \mathbf{p}' ; \varepsilon_n), \quad (16) $$

the quasiclassical transport equation for $\tilde{g}$ and the constitutive equations for the self energies. The general structure of the impurity self-energy matrix is given by

$$ \hat{\Sigma}_{\text{imp}}(\mathbf{p} ; \varepsilon_n) = \begin{pmatrix} \Sigma_{\text{imp}} + h_{\text{imp}} \cdot \sigma & \Delta_{\text{imp}} i\sigma_y + \Delta_{\text{imp}} i\sigma_y \cdot i\sigma_y \\ \Delta_{\text{imp}} i\sigma_y + \Delta_{\text{imp}} i\sigma_y \cdot \sigma & \Sigma_{\text{imp}} + h_{\text{imp}} \cdot \sigma \cdot \sigma \end{pmatrix}, \quad (17) $$
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where the same symmetry relations in Eqs. (11-12) relate the particle-hole components of the self-energy.

The matrix $\hat{u}(\hat{p}, \hat{p}')$ represents the effective impurity potential for scattering of quasiparticles from $\hat{p} \rightarrow \hat{p}'$ on the Fermi surface. In the simplest formulation of the HSM we neglect spin-flip scattering by the aerogel, assume that the aerogel is isotropic on the coherence length scale and retain only the isotropic scattering potential. The neglect of spin-flip scattering is justified for $^3\text{He}$-aerogel with a small concentration of $^4\text{He}$ added to displace the $^3\text{He}$ solid layers. The resulting scattering potential is then modeled by a single s-wave matrix element, $\hat{u}(\hat{p}, \hat{p}') = u_0 \hat{1}$.

The assumption of isotropic scattering is more questionable. However, anisotropic scattering and orientational correlations of the anisotropic scattering centers representing the silica strands can be incorporated into the HSM if needed.

The Landau molecular-field self-energy

$$\hat{\Sigma}_{\text{FL}}(\hat{p}) = \left( \begin{array}{cc} \Sigma_{\text{FL}} + \mathbf{h}(\hat{p}) \cdot \boldsymbol{\sigma} & 0 \\ 0 & \Sigma_{\text{FL}} + \mathbf{h}(\hat{p}) \cdot \sigma'' \end{array} \right),$$

is a functional of the diagonal components of the quasiclassical propagator. The scalar and vector components represent internal fields generated by the quasiparticle interactions through the disturbance of the quasiparticle spectrum in response to pairing correlations or external fields. The scalar molecular field is given by

$$\Sigma^{\text{MF}}(\hat{p}) = T \sum_{\varepsilon_n} \int \frac{d\Omega_{\hat{p}'}}{4\pi} A^s(\hat{p}, \hat{p}') \; \mathbf{g}(\hat{p}'; \varepsilon_n),$$

where $A^s(\hat{p}, \hat{p}')$ is the spin-independent quasiparticle-quasiparticle scattering amplitude. Similarly, the vector component of the molecular field

$$\mathbf{h}(\hat{p}) = T \sum_{\varepsilon_n} \int \frac{d\Omega_{\hat{p}'}}{4\pi} A^v(\hat{p}, \hat{p}') \; \mathbf{g}(\hat{p}'; \varepsilon_n),$$

represents the internal exchange field acting on quasiparticles at the position $\hat{p}$ on the Fermi surface, and is determined by the quasiparticle-quasiparticle exchange scattering amplitude, $A^v(\hat{p}, \hat{p}')$. These forward scattering amplitudes are related to the Fermi-liquid parameters, $F_{\ell}^{(s,v)}$, through the Legendre expansion, $^{17}$

$$A^{(s,v)} = \sum_{\ell} \frac{F_{\ell}^{(s,v)}}{1 + F_{\ell}^{(s,v)}/2\ell + 1} P_{\ell}(\hat{p} \cdot \hat{p}').$$

In calculating the linear response to a uniform magnetic field only the exchange interactions, $F_0^u$ and $F_2^u$, contribute for isotropic impurity scattering in the Born and unitarity scattering limits.

Finally, we have the Zeeman coupling to an external magnetic field,

$$\hat{\mathbf{e}}_{\text{ext}} = \mathbf{h}_{\text{ext}} \cdot \hat{\Sigma}, \quad \text{with} \quad \mathbf{h}_{\text{ext}} = -Z_0^u \mu \mathbf{H},$$

$$\hat{\Sigma} = \frac{i}{\hbar}(1 + \tau_3)\mathbf{\sigma} + \frac{i}{\hbar}(1 - \tau_3)\sigma'', \quad \text{and} \quad \mu = \gamma \hbar/2 \quad \text{is the nuclear magnetic moment of the $^3\text{He}$ nucleus. The factor $Z_0^u = 1/(1 + F_0^u)$ is the high-energy renormalization}$
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of the Zeeman coupling of the quasiparticle moment to the external field. Spin-
rotation invariance in normal liquid $^3$He fixes the renormalization factor in terms
of the quasiparticle exchange interaction, $F_n$.\textsuperscript{12}

The quasiclassical transport equation and constitutive equations for the self-
energies are supplemented by Eilenberger's normalization condition for the matrix
propagator,\textsuperscript{18}

$$\hat{\delta}^2 = -\pi^2 \hat{1}.$$  \hspace{1cm} (23)

In addition to the overall normalization of the propagator, the normalization condition is useful in simplifying the linearized transport equation for the response to an external field.

4.1. Homogeneous equilibrium in zero field

For the Balian-Werthamer phase the order parameter is given by

$$\hat{\Delta}(\hat{p}) = \Delta \hat{R} \cdot \hat{p},$$  \hspace{1cm} (24)

where $\hat{R}$ is the relative spin-orbit rotation matrix that defines the B-phase.\textsuperscript{19} The BW state is isotropic under joint spin and orbital rotations; as a result the magnitude of the order parameter, $|\Delta(\hat{p})| = \Delta(T)$, is also isotropic. The magnetic susceptibility is determined by exchange interactions and nonmagnetic scattering; we can safely neglect nuclear dipolar interactions, and for the purpose of calculating the susceptibility we can set $\hat{R} = 1$.

The homogeneous equilibrium solution for the propagator and self-energies of the bulk BW phase is straightforward to generalize for isotropic scattering, and has been investigated by Buchholz and Zwicknagl for the BW model for a superconductor.\textsuperscript{20} The molecular field corrections vanish in zero field, so we are left with the mean-field order parameter and impurity self energy in the homogeneous transport equation,

$$i\varepsilon_n \hat{\tau}_3 - \hat{\Sigma}_{\text{imp},0} - \hat{\Delta}_0 \cdot \hat{g}_0 = 0,$$  \hspace{1cm} (25)

where $\hat{g}_0$ is the equilibrium propagator in zero field. The solution for $\hat{g}_0$ for an isotropic BW phase with isotropic scattering has the same matrix form as that for the pure BW phase,

$$\hat{g}_0 = -\pi \frac{i\varepsilon_n \hat{\tau}_3 - \hat{\Delta}_0(\hat{p};\varepsilon_n)}{\sqrt{\varepsilon_n^2 + |\hat{\Delta}_0|^2}},$$  \hspace{1cm} (26)

where the renormalized Matsubara frequencies and order parameter are related to zero-field impurity self energy by (we drop the subscript ‘0’ unless it is explicitly needed)

$$i\varepsilon_n = i\varepsilon_n - \Sigma_3(\varepsilon_n),$$  \hspace{1cm} (27)

$$\hat{\Delta}(\hat{p};\varepsilon_n) = \Delta(\hat{p}) + \Delta_{\text{imp}}(\varepsilon_n),$$  \hspace{1cm} (28)

with the zero-field impurity self energy given by,

$$\hat{\Sigma}_{\text{imp}} = \Sigma_1(\varepsilon_n) \hat{1} + \Sigma_3(\varepsilon_n) \hat{\tau}_3 + \hat{\Delta}_{\text{imp}}(\varepsilon_n).$$  \hspace{1cm} (29)
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However, the off-diagonal impurity self energy, $\Delta_{\text{int}}(\varepsilon_n)$, vanishes for pure s-wave scattering because the driving term in the equation for $\Delta(\mathbf{p},\varepsilon_n)$ is the mean field order parameter $\langle \Delta(\mathbf{p}) \rangle$, which vanishes when averaged over the Fermi surface,\textsuperscript{21}

$$
\langle \Delta(\mathbf{p}) \rangle = \int \frac{d\Omega_{\mathbf{p}}}{4\pi} \Delta(\mathbf{p}) = 0 .
$$

(30)

Thus, $\hat{\Delta} = \Delta(\mathbf{p})$, and the t-matrix is isotropic and reduces to diagonal form,

$$
\hat{t} = t_1 \mathbb{1} + t_3 \hat{t}_3 ,
$$

(31)

$$
t_1 = \frac{\sqrt{\sigma(1 - \sigma)}}{\pi N_f} \frac{\varepsilon_n^2 + \Delta^2}{\varepsilon_n^2 + \Delta^2(1 - \sigma)} ,
$$

(32)

$$
t_3 = -\frac{\sigma}{\pi N_f} \frac{i \varepsilon_n \sqrt{\varepsilon_n^2 + \Delta^2}}{\varepsilon_n^2 + \Delta^2(1 - \sigma)} ,
$$

(33)

where $\sigma$ is the normalized scattering cross-section and is related to the scattering potential, $u_0$, and scattering phase shift, $\delta_0$, by

$$
\sigma = \sin^2 \delta_0 = \frac{(\pi N_f u_0)^2}{1 + (\pi N_f u_0)^2} .
$$

(34)

The unitarity limit corresponds to strong scattering, $u_0 \to \infty$, in which case $\sigma \to 1$ ($\delta_0 \to \pi/2$). Born scattering is the limit of small cross section, $\sigma \ll 1$, ($\delta \ll 1$). The other key parameter needed to relate the HSM parameters to the properties characterizing the aerogel is the mean free path, $\ell$. In the HSM the mean-free path is determined by the Fermi velocity and the mean time for a quasiparticle to scatter off an impurity, $\ell = v_F \tau$. The scattering time is determined by the imaginary part of the normal-state impurity self energy,

$$
\Gamma_N = \frac{h}{2\tau} = \frac{n_{\text{imp}}}{\pi N_f} \sigma .
$$

(35)

Thus, the isotropic HSM is defined by two parameters which we take to be the mean free path, $\ell$, and identify with the geometric mean free path for a specific aerogel: and the dimensionless cross-section of the scattering centers, $\sigma$, which can vary from $\sigma = 0$ (Born limit) to $\sigma = 1$ (unitarity scattering limit). Since the geometric cross-section of a typical aerogel strand is large compared to the Fermi wavelength the unitarity limit is more appropriate for modeling the effects of scattering by aerogel.

Finally, the renormalized Matsubara frequencies are given by the solution to the equation.

$$
\varepsilon_n = \varepsilon_n + \Gamma_N \frac{\varepsilon_n \sqrt{\varepsilon_n^2 + \Delta^2}}{\varepsilon_n^2 + \Delta^2(1 - \sigma)} .
$$

(36)

The equilibrium equations in zero field are closed by the self-consistency equation for the order parameter:

$$
\Delta(\mathbf{p}) = \frac{\pi T}{|\varepsilon_n|} \sum_{\varepsilon_n} \int \frac{d\Omega_{\mathbf{p}'}}{4\pi} V^4(\mathbf{p} \cdot \mathbf{p}') \frac{\Delta(\mathbf{p}')}{\sqrt{\varepsilon_n^2 + \Delta^2}} ,
$$

(37)
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which reduces to the BCS gap equation,

$$\Delta = V_1 \pi T \sum_{\varepsilon_n}^{\mid\varepsilon_n\mid \leq \varepsilon_c} \frac{\Delta}{\sqrt{\varepsilon_n^2 + \Delta^2}}.$$  \hspace{1cm} (38)

The linearized gap equation for pure $^3$He-B ($\varepsilon_n \rightarrow \varepsilon_n$) determines the bulk transition temperature.

$$\frac{1}{V_1} = \pi T c_0 \sum_{\varepsilon_n}^{\mid\varepsilon_n\mid \leq \varepsilon_c} \frac{1}{\mid\varepsilon_n\mid} = K(T_{c0}) \simeq \ln(1.13\varepsilon_c/T_{c0}).$$  \hspace{1cm} (39)

Using Eq. (39) we can remove the pairing interaction and cutoff in favor of the bulk transition temperature, $T_{c0}$. Thus, the gap equation for $^3$He-B-aerogel becomes,

$$\ln(T/T_{c0}) = \pi T \sum_{\varepsilon_n} \left( \frac{1}{\sqrt{\varepsilon_n^2 + \Delta^2}} - \frac{1}{\mid\varepsilon_n\mid} \right).$$  \hspace{1cm} (40)

The linearized gap equation reduces to the Abrikosov-Gorkov formula\textsuperscript{22} for the suppression of $T_c$ by scattering in the HSM for the aerogel,\textsuperscript{10}

$$\ln(T_c/T_{c0}) = 2\pi T_c \sum_{n \geq 0} \left( \frac{1}{\varepsilon_n + \Gamma N} - \frac{1}{\varepsilon_n} \right).$$  \hspace{1cm} (41)

Density of States

In superconductors with an order parameter that breaks orbital rotation symmetry, such as d-wave pairing in the cuprates or f-wave pairing in UPt$_3$, scattering from an impurity, particularly in the strong scattering limit, or from a surface, leads to the formation of quasiparticle states at the Fermi level that are bound to the impurity, or surface, within a distance of order the coherence length.\textsuperscript{23,24} In the case of a random distribution of scattering centers the impurity bound states at the Fermi level broaden into an impurity band near the Fermi level.\textsuperscript{25} The same is true for the p-wave phases of $^3$He in aerogel.

The equilibrium solutions for the quasiclassical propagator also determine the quasiparticle excitation spectrum below $T_c$. Continuation of the Matsubara Green’s functions to real energies determines the retarded equilibrium Greens functions,

$$\tilde{g}^R(\hat{p}; \varepsilon) = \tilde{g}(\hat{p}; i \varepsilon_n \rightarrow \varepsilon),$$  \hspace{1cm} (42)

which provide spectral information for the low-energy quasiparticle states in the presence of pairing correlations and disorder. In particular the angle-resolved local density of states, averaged over spin projections, is given by

$$N(\hat{p}; \varepsilon) = -\frac{1}{\pi} \text{Im} g^R(\hat{p}; \varepsilon),$$  \hspace{1cm} (43)

whereas the spin-polarization spectral density is given by

$$P(\hat{p}; \varepsilon) = -\frac{1}{\pi} \text{Im} g^R(\hat{p}; \varepsilon).$$  \hspace{1cm} (44)
Figure 3 shows the self-consistently determined density of states (DOS) for $^3$He-B-aerogel calculated for $\ell = 1800 \text{Å}$ and $\sigma = 0.99$ as a function of energy for pressures spanning the range $P = 5 - 30$ bar. The pairbreaking parameter

$$\alpha = \frac{\hbar}{2\pi \tau T_c_0} = \frac{\xi_0}{\ell},$$

(45)

decreases from $\alpha = 0.23 \rightarrow 0.09$ over the same pressure range. Note that even at the highest pressure, where the pairbreaking effect is weakest, there is a substantial density of quasiparticle states at the Fermi level from the impurity-induced Andreev bound states. At lower pressures the B-phase becomes completely gapless. These quasiparticle states lead to an increase in the spin susceptibility. Their polarizability is described by the polarization spectral density. In particular, the change in magnetization can be expressed in terms of $\mathbf{P}(\hat{\mathbf{p}}, \varepsilon)$ by analytic continuation of Eq.(5) to real energies,

$$\mathbf{m} = 2 \int_{-\infty}^{+\infty} d\varepsilon f(\varepsilon) \langle \mathbf{P}(\hat{\mathbf{p}}, \varepsilon) \rangle,$$

(46)

where $f(\varepsilon)$ is the Fermi function.

### 4.2. Linear Response to the Nuclear Zeeman coupling

The linear response of $^3$He-aerogel to the nuclear Zeeman field is obtained by linearizing the propagator in the external field

$$\hat{g} = \hat{g}_0 + \hat{g}_1 + \ldots,$$

(47)

and similarly for the order parameter and self-energies; $\hat{g}_0$ is the zero-field propagator given in Eq. (26). The first-order correction, $\hat{g}_1$, satisfies the linearized
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transport equation,

$$\left[ i \varepsilon_n \hat{\tau}_3 - \hat{\Sigma}_{\text{imp},0} - \hat{\Delta}_0 , \hat{\mathbf{g}}_1 \right] = \left[ \hat{\nu}_{\text{ext}} + \hat{\Sigma}_1 + \hat{\Delta}_1 , \hat{\mathbf{g}}_0 \right], \quad (48)$$

and the corresponding linearized normalization conditions,

$$\hat{\mathbf{g}}_0 \hat{\mathbf{g}}_1 + \hat{\mathbf{g}}_1 \hat{\mathbf{g}}_0 = 0, \quad \text{with} \quad \hat{\mathbf{g}}_0^2 = -\pi^2 \mathbf{1}. \quad (49)$$

The operators on the left-side of Eq. (48) can be expressed in terms of $\hat{\mathbf{g}}_0$,

$$i \varepsilon_n \hat{\tau}_3 - \hat{\Sigma}_{\text{imp},0} - \hat{\Delta}_0 = N_0^{-1} \hat{\mathbf{g}}_0 - n_{\text{imp}} t_1 \mathbf{1}, \quad (50)$$

with $N_0 = -\pi / \sqrt{\varepsilon_n^2 + \Delta_0^2}$. Thus, Eq. (48) becomes,

$$[\hat{\mathbf{g}}_0 , \hat{\mathbf{g}}_1] = N_0 \left[ \hat{\nu}_{\text{ext}} + \hat{\Sigma}_1 + \hat{\Delta}_1 , \hat{\mathbf{g}}_0 \right]. \quad (51)$$

Combining Eq. (51) with the normalization conditions in Eq. (49) gives the linear correction to the matrix propagator in terms of the external field, first-order self-energies and zero-field propagator,

$$\hat{\mathbf{g}}_1 = \frac{1}{2 \pi \sqrt{\varepsilon_n^2 + \Delta_0^2}} \left\{ \pi^2 \left( \hat{\nu}_{\text{ext}} + \hat{\Sigma}_1 + \hat{\Delta}_1 \right) + \hat{\mathbf{g}}_0 \left( \hat{\nu}_{\text{ext}} + \hat{\Sigma}_1 + \hat{\Delta}_1 \right) \hat{\mathbf{g}}_0 \right\}. \quad (52)$$

The detailed solution for $\hat{\mathbf{g}}_1$ is most conveniently represented in terms of components of $\hat{\mathbf{g}}_1$ which are symmetrized between the particle and hole components, i.e. for any particular component of $\hat{\mathbf{g}}_1$ define $X^\pm \equiv X \pm \Sigma$. The symmetry relations (Eqs. 11-12) imply that $X^\pm$ will have even or odd symmetry with respect to $(\mathbf{p}, \varepsilon_n) \rightarrow (-\mathbf{p}, -\varepsilon_n)$ or $(\mathbf{p}, \varepsilon_n) \rightarrow (-\mathbf{p}, \varepsilon_n)$.

The self-consistent solution for $\hat{\mathbf{g}}_1$ requires the first-order corrections for the internal field, the order parameter, and self-energies. We can express the correction to the impurity self-energy in terms of the first-order correction to the t-matrix,

$$\hat{t}_1 = N_f \hat{t}_0 (\hat{\mathbf{g}}_1) \hat{t}_0, \quad (53)$$

where $\langle \ldots \rangle$ denotes the average over the Fermi surface. Equations (31-33) and (52) for $\hat{t}_0$ and $\hat{g}_1$ determine the first-order corrections to the impurity self-energy, which have the general structure given in Eq. (17). In terms of the symmetrized components, the linear corrections to the impurity self-energy have the form,

$$\Sigma_{\text{imp},1}^+ = n_{\text{imp}} N_f \left\{ (t_1^2 + t_2^2) \left( g_1^+ \right) + 2 t_1 t_3 \left( g_i^+ \right) \right\}, \quad (54)$$

$$\Sigma_{\text{imp},1}^- = n_{\text{imp}} N_f \left\{ (t_1^2 + t_2^2) \left( g_1^- \right) + 2 t_1 t_3 \left( g_i^- \right) \right\}, \quad (55)$$

$$h_{\text{imp},1}^+ = n_{\text{imp}} N_f \left\{ (t_1^2 + t_2^2) \left( g_i^+ \right) + 2 t_1 t_3 \left( g_1^+ \right) \right\}, \quad (56)$$

$$h_{\text{imp},1}^- = n_{\text{imp}} N_f \left\{ (t_1^2 + t_2^2) \left( g_i^- \right) - 2 t_1 t_3 \left( g_1^- \right) \right\}, \quad (57)$$

$$\Delta_{\text{imp},1}^\pm = n_{\text{imp}} N_f \left( t_1^2 - t_2^2 \right) \left( f_1^\pm \right), \quad (58)$$

$$\Delta_{\text{imp},1}^\pm = n_{\text{imp}} N_f \left( t_1^2 - t_2^2 \right) \left( f_i^\pm \right). \quad (59)$$

The key corrections are the internal fields, $h_{\text{imp},1}^\pm$, and the impurity correction to the order parameter, $\Delta_{\text{imp},1}^\pm$. The scalar corrections, $\Sigma_{\text{imp},1}^\pm$, turn out to vanish.
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The external field also induces an internal exchange field given by Eq. (20). Thus, the total effective field is

$$\hat{h}^+(\hat{p}; \varepsilon_n) = h^+_{\text{imp},1}(\varepsilon_n) + h^+_{\text{FL},1}(\hat{p}) + 2h_{\text{ext}}, \quad (60)$$

$$\hat{h}^-(\hat{p}; \varepsilon_n) = h^-_{\text{imp},1}(\varepsilon_n) + h^-_{\text{FL},1}(\hat{p}) \, , \quad (61)$$

with the exchange terms given by

$$h^\pm_{\text{FL},1}(\hat{p}) = \int \frac{d^df'}{4\pi} A^\mu(\hat{p} \cdot \hat{p}') T \sum_{\varepsilon_n} g^\pm_1(\hat{p}', \varepsilon_n) \cdot (62)$$

In general there is also a linear correction to the spin-triplet order parameter,

$$\Delta^\pm(\hat{p}) = \int \frac{d^df'}{4\pi} V^\mu(\hat{p} \cdot \hat{p}') T \sum_{\varepsilon_n} f^\pm_1(\hat{p}', \varepsilon_n) \cdot (63)$$

which may be combined with the off-diagonal impurity correction, $\Delta^\pm(\hat{p}; \varepsilon_n) = \Delta^\pm_1(\hat{p}; \varepsilon_n) + \Delta^\pm_{\text{imp},1}(\varepsilon_n)$. The full solution for $g_1$ can be expressed in terms of these effective fields:

$$g^+_1 = 0 \, , \quad (64)$$

$$g^-_1 = \frac{\pi}{(\varepsilon_n^2 + \Delta^2)^{3/2}} i\varepsilon_n \left(\Delta \cdot \Delta^+_1\right) \, , \quad (65)$$

$$g^+_1 = \frac{\pi}{(\varepsilon_n^2 + \Delta^2)^{3/2}} \Delta(\Delta \cdot \hat{h}^+) \, , \quad (66)$$

$$g^-_1 = \frac{\pi}{(\varepsilon_n^2 + \Delta^2)^{3/2}} \left\{ -\Delta \times (\Delta \times \hat{h}^-) + \varepsilon_n \Delta \times \hat{h}^- \right\} \, , \quad (67)$$

$$f^+_1 = \frac{\pi}{(\varepsilon_n^2 + \Delta^2)^{3/2}} (-i\varepsilon_n) \Delta \cdot \hat{h}^+ \, , \quad (68)$$

$$f^-_1 = 0 \, , \quad (69)$$

$$f^+_1 = \frac{\pi}{\sqrt{\varepsilon_n^2 + \Delta^2}} \frac{\pi}{(\varepsilon_n^2 + \Delta^2)^{3/2}} \Delta(\Delta \cdot \hat{h}^+) \, , \quad (70)$$

$$f^-_1 = \frac{\pi}{\sqrt{\varepsilon_n^2 + \Delta^2}} \frac{\pi}{(\varepsilon_n^2 + \Delta^2)^{3/2}} \left\{ -\varepsilon_n \Delta \times \hat{h}^- + \Delta \times (\Delta \times \hat{h}^-) \right\} \cdot (71)$$

We can further simplify the results by noting that $\Delta^+_1$ obeys homogeneous, linear self-consistency equations which allow only the solution $\Delta^+_1 = 0$. Thus, $g^-_1$ and $f^+_1$ also vanish.

The magnetization of $^3$He-B in aerogel can be calculated from the Fermi surface average of Eq. (66) and the corresponding total field. Thus, the susceptibility (Eq. (4)) is given by

$$\frac{\chi_B}{\chi_N} = 1 + \frac{1}{2\mu_B} T \sum_{\varepsilon_n} \langle g^+_1 \rangle \cdot (72)$$

Note that only the even (+) part of $g$ contributes to the magnetization. We can calculate the change in the magnetization,

$$m = T \sum_{\varepsilon_n} \langle g^+_1 \rangle \, , \quad (73)$$
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from the solutions for $g_1 \pm$ in Eqs. (66-67), and the self-consistency equations for $h^\pm$ - Eqs. (56,57,62) - and $\Delta_1$ given in Eq. (59) and Eq. (71). Inserting the result for $g_1^+$ from Eq. (66) gives,

$$m = T \sum_{\varepsilon_n} K^+(\varepsilon_n) \langle \hat{p} \cdot \hat{h}^+ \rangle, \quad (74)$$

where

$$K^+(\varepsilon_n) \equiv \frac{\pi \Delta^2}{[\varepsilon_n^2 + \Delta^2]^{3/2}}. \quad (75)$$

The Fermi surface average of the external field and the exchange contributions to the total field can be evaluated, or expressed in terms of $m$,

$$\langle \hat{p} \cdot \hat{h}^+ \rangle = \frac{2}{3} h_{\text{ext}} + \frac{1}{3} A_0^+ m + \frac{2}{15} A_2^+ m + \frac{1}{3} h_{\text{imp}}^+. \quad (76)$$

We then obtain an equation for $m$ that depends on the impurity contribution to the internal field,

$$\left(1 - \frac{1}{3} A_0^+ y_\parallel - \frac{2}{15} A_2^+ y_\parallel \right) m = \frac{2}{3} y_\parallel h_{\text{ext}} + \frac{1}{3} T \sum_{\varepsilon_n} K^+(\varepsilon_n) h_{\text{imp}}^+, \quad (77)$$

where

$$y_\parallel = T \sum_{\varepsilon_n} K^+(\varepsilon_n) \quad (78)$$

is related to the Yosida function ($Y = 1 - y_\parallel$) evaluated with renormalized Matsubara frequencies. To complete the calculation we need to calculate the sum over the impurity correction to the internal field, $h_{\text{imp}}^+$. In general, for scattering that is neither in the Born or unitarity limits, the self-consistent solution for the impurity field entering Eq. (77) involves both the even- and odd components of the quasiparticle polarization, $g_1^\pm$ (see Eq. (56)). However, for the Born and unitarity scattering limits the coupling to $g_1^-$ drops out, and an analytic result can be obtained for $h_{\text{imp}}^+(\varepsilon_n)$.

In the unitarity scattering limit, $\bar{\sigma} \to 1$, in which case $t_1 \to 0$ and

$$t_3 \to \frac{1}{\pi N_f} \sqrt{\frac{\varepsilon_n^2 + \Delta^2}{\varepsilon_n}}. \quad (79)$$

Thus, the equation for $h_{\text{imp}}^+(\varepsilon_n)$:

$$h_{\text{imp}}^+ = -\frac{1}{2\pi \tau} \left(\varepsilon_n^2 + \frac{\Delta^2}{\varepsilon_n^2}\right) K^+(\varepsilon_n) \langle \hat{p} \cdot \hat{h}^+ \rangle, \quad (80)$$

depends on the same Fermi surface average that determines $m$ in Eq. (74), in which case we can solve for $h_{\text{imp}}^+$, and then $T \sum_{\varepsilon_n} K^+(\varepsilon_n) h_{\text{imp}}^+(\varepsilon_n)$, in terms of $h_{\text{ext}}$ and $m$:

$$h_{\text{imp}}^+ = -\frac{1}{2\pi \tau} \left(\varepsilon_n^2 + \frac{\Delta^2}{\varepsilon_n^2}\right) K^+ \left[\frac{2}{3} h_{\text{ext}} + \frac{1}{3} A_0^+ m + \frac{2}{15} A_2^+ m \right]. \quad (81)$$
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The result for the change in magnetization reduces to

$$m = \frac{\frac{2}{3} y_{3/2} \mathbf{h}_{\text{ext}}}{\left[ 1 - \frac{1}{3} A_0^a y_{3/2} - \frac{2}{15} A_2^a y_{3/2} \right]}, \quad (82)$$

where the impurity-renormalized response function is given by

$$\tilde{y}_{3/2} = \pi T \sum_{\varepsilon_n} \frac{\Delta^2}{[\varepsilon_n^2 + \Delta^2]^{3/2}} \left\{ \frac{1}{1 + \frac{1}{6\tau} \left( \frac{\Delta^2}{\varepsilon_n^2} \right) \sqrt{\varepsilon_n^2 + \Delta^2}} \right\} \text{ (unitarity)}. \quad (83)$$

Thus, the susceptibility of the BW phase of $^3$He in aerogel in the HSM is obtained from Eqs. (72), (73) and (82),

$$\frac{\chi_B}{\chi_N} = 1 - \left( \frac{1}{1 + F_0^a} \right) \left( 1 - \left( \frac{1}{3} A_0^a + \frac{2}{15} A_2^a \right) \tilde{y}_{3/2} \right). \quad (84)$$

This result can be expressed in terms of the conventional Landau parameters, $F_0^a$ and $F_2^a$.

$$\frac{\chi_B}{\chi_N} = \frac{(1 + F_0^a) \left[ \frac{2}{3} + \tilde{Y} \left( \frac{1}{3} + \frac{1}{A_2^a} \right) \right]}{1 + F_0^a \left( \frac{2}{3} + \tilde{Y} \right) + \frac{1}{5} F_2^a \left( \frac{2}{3} + \tilde{Y} \right) + \frac{1}{5} F_2^a F_0^a \tilde{Y}}, \quad (85)$$

where $\tilde{Y} = 1 - \tilde{y}_{3/2}$ is the impurity renormalized Yosida function given in Eq. (7).

**Born limit**

In the Born limit, $\tilde{\sigma} \to 0$, we can carry through essentially the same analysis as in the unitarity case to calculate the impurity contribution to the local field and obtain.

$$\mathbf{h}_{\text{imp}} = \frac{1}{2\pi \tau} \left( \frac{K^+}{1 - \frac{1}{n_{\sigma \sigma}} K^+} \right) \left[ \frac{2}{3} \mathbf{h}_{\text{ext}} + \left( \frac{1}{3} A_0^a + \frac{2}{15} A_2^a \right) \mathbf{m} \right]. \quad (86)$$

For the Born limit the susceptibility is again given by (84), but with

$$\tilde{y}_{3/2} = \pi T \sum_{\varepsilon_n} \frac{\Delta^2}{[\varepsilon_n^2 + \Delta^2]^{3/2}} \left\{ \frac{1}{1 - \frac{1}{6\tau} \left[ \frac{\Delta^2}{\varepsilon_n^2 + \Delta^2} \right]^{3/2}} \right\} \text{ (Born)}, \quad (87)$$

which yields Eq. (6).

**4.3. Ginzburg-Landau Limit**

For temperatures just below $T_c$ the mean-field spin susceptibility for non-ESF states decreases linearly with temperature. The temperature interval over which the susceptibility is linear defines the region of applicability of the Ginzburg-Landau (GL) theory. The general result for the susceptibility of the B phase in the GL
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region is expressed in terms of the GL material parameter for the change in the Zeeman energy, i.e. $\Delta \Omega_{\text{Zeeman}} = g_z \Delta^2 H^2, \omega_0$ which implies

$$\chi_B - \chi_N = -2 g_z \Delta(T)^2 \sim -g_z T_c (T_c - T).$$  \hspace{1cm} (88)

We can obtain the GL limit from the general result for $\chi_B$ and identify the GL material parameter $g_z$ for the HSM of $^3$He-aerogel. For $T \to T_c$, the spin susceptibility reduces to

$$\frac{\chi_B}{\chi_N} = 1 - \frac{1}{3} \left( \frac{1}{1 + F_0^2} \right) \tilde{y}_3^3,$$  \hspace{1cm} (89)

with

$$\tilde{y}_3^3 \to \left( \frac{\Delta(T)}{2 \pi T_c} \right)^2 \sum_{n \geq 0} \left( \frac{1}{n + \frac{1}{2} + \frac{1}{2} \frac{1}{2 \pi T_c}} \right)^3.$$  \hspace{1cm} (90)

Thus, the spin susceptibility reduces to

$$\frac{\chi_B}{\chi_N} = 1 - \frac{1}{3} \left( \frac{1}{1 + F_0^2} \right) \left( \frac{\Delta(T)}{2 \pi T_c} \right)^2 S_3(x),$$  \hspace{1cm} (91)

where

$$S_m(x) = \sum_{n \geq 0} \left( \frac{1}{n + \frac{1}{2} + x} \right)^m, \hspace{1cm} m > 1,$$  \hspace{1cm} (92)

and $x = 1/4 \pi T_c$ is the pairbreaking parameter. The order parameter in the GL limit can also be expressed in terms of $T_c$ and the sums defined in Eq. (92),

$$\Delta \to \sqrt{8 \pi^2 \left( \frac{1 - x S_2(x)}{S_3(x) + (2x - 1) x S_4(x)} \right)} T_c \sqrt{1 - T/T_c}.$$  \hspace{1cm} (93)

The result for the GL material parameter that determines the change in spin susceptibility is then identified as

$$g_z = \frac{N_f \mu^2}{(1 + F_0^2)^2} \frac{S_3(x)}{12 \pi^2 T_c^2}.$$  \hspace{1cm} (94)

For $x = 0$ we recover the known weak-coupling result for $g_z$ for pure $^3$He-B.

$$g_z^{\text{pure}} = \frac{N_f \mu^2}{(1 + F_0^2)^2} \frac{7 \zeta(3)}{12 \pi^2 T_c^2}.$$  \hspace{1cm} (95)

5. Susceptibility

In Figure 4 we show the impurity-renormalized Yosida function, $\tilde{Y}$ at $\nu = 10$ bar in the unitarity limit of the HSM for mean-free paths of $\ell = 1700 \text{ Å}$, $\ell = 5000 \text{ Å}$ and $\ell \to \infty$. The increase in $\tilde{Y}(T = 0)$ reflects the density of fermionic states below the gap that can contribute to the low-field magnetization. We also show the result for $Y$ in the Born scattering limit at $\ell = 1700 \text{ Å}$. The smaller cross-section is a less effective pairbreaker, which is reflected in the stronger condensate suppression of the spin-polarization response function.
Fig. 4. The renormalized Yosida function for the spin response function for the BW state vs. $T/T_{cs}$ for the unitarity limit (solid lines) of the HSM for $P = 10$ bar and $\ell = 1700 \text{ Å}, \ell = 5000 \text{ Å}$ and $\ell = \infty$ (bulk). The result for the Born limit (dashed line) is shown for $\ell = 1700 \text{ Å}$.

Fig. 5. The susceptibility for the BW state vs. $T/T_{cs}$ scaled to $\chi_N$ in the unitarity limit (solid lines) of the HSM for $P = 10$ bar and $\ell = 1700 \text{ Å}, \ell = 5000 \text{ Å}$ and $\ell = \infty$ (bulk). The GL slope for $\ell = 1700 \text{ Å}$ is plotted to indicate the limited range of validity of the GL result. Also shown for $\ell = 1700 \text{ Å}$ is the effect of “switching off” the impurity correction to the local field in the unitarity limit (dotted curve). The Born limit for $\chi_B$ is shown as the dashed curve for $\ell = 1700 \text{ Å}$. The material parameters used for $P = 10$ bar are: $v_f = 45.66 \text{ m/s}, T_{c0} = 1.83 \text{ mK}, F_0^a = -0.744, F_2^a = -0.466$.

The magnetic susceptibility for $^3$He-B in aerogel is shown in Fig. 5 for the same pressure, scattering cross-sections and mean-free paths. The Ginzburg-Landau limit for unitarity scattering and $\ell = 1700 \text{ Å}$ is also shown. The Fermi-liquid parameters are taken from the tables in the review article by Halperin and Varoquaux. The effect of scattering by the aerogel is substantial; $T_c$ is suppressed to $T_c \approx 0.5T_{c\ell}$.
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at $P = 10$, bar while the susceptibility is increased by nearly a factor of 2 at low temperatures. Another feature to note is that the Ginzburg-Landau limit for $\chi_B$ is valid only in a limited range of temperatures below $T_c$; thus, linear fits to limited data for $\chi(T)$ may lead to misleading information on the material parameters when compared with the GL result.

Fig. 6. The density of states for the BW state vs. energy, $\varepsilon$, for both the unitarity and Born scattering limits in the HSM for $P = 10$ bar and $\ell = 1800$ Å

Fig. 7. The spin polarization spectral function for the BW state vs. energy, $\varepsilon$, for both the unitarity and Born scattering limits in the HSM for $P = 10$ bar and $\ell = 1800$ Å

Also shown in Fig. 5 for $\ell = 1700$ Å is a comparison between the unitarity and Born limits for the low temperature susceptibility. Unitary scattering is more effective
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at pairbreaking than Born scattering. In the same figure we show the effect of the impurity vertex correction to the internal field (Eq. 80)) on the spin susceptibility at $\ell = 1700\,\text{Å}$ in the unitarity limit; the impurity field leads to a relatively small correction compared to the overall increase in the susceptibility that results from the polarizability of the quasiparticle states. The large density of states near zero energy (see Fig. 6), characteristic of the unitarity limit, contributes relatively little to the polarization, as is evident from Fig. 7. As a result the quasiparticles states at finite energy below and near the gap edge, which are present for both unitarity and Born scattering are mainly responsible for the increase in the spin susceptibility. The integrated polarization from the gapless states below the Fermi level is larger in the unitarity limit reflecting the stronger pairbreaking effect for scattering from larger cross-section impurities.

5.1. Comparison with Existing Data for the Susceptibility

Experimental data for the nuclear spin susceptibility of $^3$He in aerogel is reported by Sprague, et al. for $P = 18.7\,\text{bar}$ and by Barker, et al. for $P = 32\,\text{bar}$. In both experiments the susceptibility was reported to decrease below $T_{\text{cu}}$ when several monolayers of $^4$He were added to suppress the solid-layer Curie susceptibility.

![Fig. 8. Magnetic susceptibility of $^3$He in aerogel at $P = 18.7\,\text{bar}$ from Sprague, et al. The bulk susceptibility data (open circles) is also shown for the same pressure. The Landau parameter $F_{\text{n}} = -0.757$ is taken from the normal state susceptibility, while $F_{\text{S}}^2$ is used to fit the bulk data to the theoretical result. The fit parameter is: $F_{\text{S}}^2 = 2.2$. The theoretical results for $^3$He-aerogel are shown for the same pressure and Landau parameters for mean free paths of $\ell = 1700, 1800, \text{and } 2000\,\text{Å}$, and $\sigma = 1$.

In Fig. 8 we compare the susceptibility data at $P = 18.7\,\text{bar}$ with the theoretical result calculated for the same pressure in the unitarity scattering limit. The comparison is only partly satisfactory. The pure $^3$He-B susceptibility reported for the same pressure by Sprague, et al. does not agree precisely with the theoretical
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result using the Fermi-liquid parameters based on the susceptibility measurements of Hoyt, et al.\textsuperscript{10} However, if we fix $F_0^1$ by the normal state susceptibility, use the Landau parameter $F_0^2$ to fit the bulk susceptibility data shown in Fig. 8, calculate the susceptibility for $^3$He-aerogel with these Landau parameters, then the comparison with the data for $^3$He-aerogel is qualitatively correct for reasonable values of the mean free path.

The comparison between the HSM for $^3$He-B in aerogel and the data of Barker et al. is slightly improved. The theoretical calculation shown in Fig. 9 was obtained with the bulk Fermi-liquid data from Halperin and Varoquaux’s tables, and a mean-free path of $\ell = 1800\,\text{Å}$ for unitarity scattering. The data of Barker, et al. agrees semi-quantitatively with the theoretical result, but the data span a limited temperature range and shows an unphysical upturn at lower temperatures, which is also visible in the data of Sprague, et al. even for the bulk measurements.

![Graph](image)

Fig. 9. Magnetic susceptibility of $^3$He in aerogel at $P = 32\,\text{bar}$ from Barker et al. The theoretical result calculated for the same pressure, a mean free path of $\ell = 1800\,\text{Å}$ for $\tilde{\sigma} = 1$.

6. Conclusion

Experiments based on NMR and acoustic attenuation indicate that the equilibrium phase of superfluid $^3$He-aerogel at all pressures is a non-ESP phase, generally assumed to be the Balian-Wertham state modified by scattering off the aerogel structure. For $^3$He-aerogel impurity scattering leads to substantial changes in the
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susceptibility of the BW phase. The available data is in semi-quantitative agreement with theoretical predictions. Additional measurements could prove important for a more definitive identification of the equilibrium order parameter, as well as for refining the theoretical model to describe the effects of disorder and scattering on the properties of superfluid $^3$He.

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