Josephson Weak Links in $^3$He and in Unconventional Superconductors

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We discuss the effects of depletion layers, interface roughness, and of spin–active barriers on the Josephson coupling of systems with unconventional pairing.

1 INTRODUCTION

Josephson effects in $^3$He and in unconventional superconductors have drawn both experimental and theoretical attention in the past few years. The large coherence length of superfluid $^3$He as compared to $^4$He is a favourable condition for getting weak links with a single-valued current-phase relation. Hence, $^3$He holds promise for the observation of Josephson’s original AC-effect in a quantum liquid. Josephson effects in junctions with heavy–Fermion superconductors have been thought to give quite definite insight into the spin and orbital symmetries of the presumably unconventional order parameter of these materials.

In general, Josephson weak links in systems with unconventional pairing (p-wave pairing, triplet pairing, etc.) are expected to show a richer variety of phenomena than conventional Josephson junctions. Different orientations of an anisotropic order parameter, for instance, will lead to Josephson couplings of different strengths, and might allow modulation of the critical current of a junction. There has also been speculation about magnetic Josephson currents in triplet superfluids.

We concentrate in this report on three selected aspects of Josephson weak links, which are of particular interest for superfluid $^3$He and unconventional superconductors. First, because of surface pairbreaking, a weak link separating two bulk systems is always accompanied by layers of size $\xi$ of a strongly distorted order parameter. Such layers affect in a characteristic way the magnitude and temperature dependence of Josephson’s critical current. Secondly, we expect conventionally paired systems to react more sensitively to microscopic imperfections (roughness) of the weak link. This effect might be relevant, for example, for an S–N–S junction formed by a dirty normal metal (N) and two superconductors (S) with one or both of them of unconventional type. Finally, we comment on the role of a spin-active interface on the Josephson coupling of two superconductors with even and odd parity pairing. Such Josephson contacts have been discussed repeatedly because of their potential use for discriminating between conventional and unconventional pairing.

2 CALCULATIONAL FRAMEWORK

Microscopic calculations of properties of superconducting or superfluid weak links (tunnel junctions, bridges, constricted channels, etc.) commonly require the most powerful techniques of the theory of superconductivity. The problems to be solved are of considerable complexity, mainly because weak-link structures are inherently spatially inhomogeneous, quantum effects dominate, and because most of the phenomena of interest occur in junctions driven into a nonequilibrium state. Probably the most efficient and comprehensive technique for this kind of problem is the ‘quasiclassical method’ developed by Eilenberger [1], Larkin and Ovchinikov [2–4] and Eliashberg [5]. Some early applications of this method to weak links with conventional superconductors are reviewed by Likharev [6]. Other representative publications on this subject are found in refs. [7–9], to mention a few.

The quasiclassical theory was readily generalized to systems with unconventional pairing (superfluid $^3$He, unconventional superconductors). Reviews on the quasiclassical approach to $^3$He have been published by Eckern [10], and Serene and Rainer [11]. The central quantities of the theory are the ‘quasiclassical propagators’ $\hat{g}(s,\vec{r},t)$ which are 4x4 Nambu matrices representing generalized density operators for quasiparticles. The propagators depend on position (\vec{r}), time (t), the excitation energy (\epsilon) of quasiparticles, and on a two-dimensional Fermi–surface variable s. The variable s is usually replaced in isotropic systems by the momentum direction \hat{p}. The quasiclassical propagators satisfy transport–like equations (Eilenberger’s equations)

$$[\epsilon \partial_t - \hat{\delta}(s,\vec{r},\epsilon,t),\hat{g}(s,\vec{r},\epsilon,t)]_{\hat{g}} + \text{i} \hat{\nabla}_s \cdot \vec{V}_s(s,\vec{r},\epsilon,t) = 0$$

and are normalized according to

$$\int_{s} \hat{g}(s,\vec{r},\epsilon,t) \hat{g}(s,\vec{r},\epsilon,t) = -n^2$$

Here, $\hat{V}_s = \hat{V}_s(s)$ is the Fermi velocity at point s on the Fermi surface, and \hat{\delta} comprises external potentials and self–energies, in particular the order parameter matrix $\hat{\Delta}(s,\vec{r},\epsilon,t)$. The self–energies have to be calculated self–consistently in terms of $\hat{g}$. We follow the notation of ref. [11], and use $\hat{g}$ and $\hat{\delta}$ without superscripts as the generic symbols for propagators and self–energies of the various kinds ($\hat{g}_{\phi}$, $\hat{g}_{\phi}$, etc.). The operation $\hat{\otimes}$ stands for a combination of conventional 4x4 matrix multiplication, the following operation in the energy and time variables

$$f(\epsilon,t) \hat{g}(\epsilon,t) = f(\epsilon \frac{\partial}{\partial \epsilon} t_1 + \frac{\partial}{\partial t_1} \epsilon, \epsilon^0 + \frac{\partial}{\partial \epsilon} t_2, t_2), |_{t_1 = \epsilon_0 = t_2}$$

and a specific assignment of Keldysh indices:

$$(f \hat{g})_{A,R} = (\hat{g})_{A,R}, (f \hat{g}) = (\hat{g})$$

(4)
The symbol \( [\hat{f},\hat{g}] \) denotes the commutator \( \hat{f} \hat{g} - \hat{g} \hat{f} \).

In order to complete the quasiclassical theory, one has to impose boundary conditions on the propagators \( \hat{g} \) at surfaces, interfaces (metallic contacts), and tunneling barriers. Of course, such boundary conditions are of central importance in a discussion of Josephson weak links, in particular for systems with unconventional pairing. At present, we have no general theory of these boundary conditions. Various plausible model boundary conditions have been suggested recently in refs. [9,12–17]. A brief review on boundary conditions for quasiclassical propagators was published by Kurkijärvi et al. [18]. Some of these boundary conditions will be discussed in section 3, where we present selected applications of the quasiclassical theory.

Equations (1) and (2), together with a set of self-consistency conditions for the self-energies, and the boundary conditions constitute the most general version of the quasiclassical theory. The theory can be simplified considerably in several limits, in which one can take advantage of additional small expansion parameters. Examples are: thermal equilibrium, the local equilibrium limit (hydrodynamic limit), the limit of low-frequency dynamics, valid at scales much longer than the inverse gap frequency, or the Ginzburg–Landau limit near \( T_c \). We refer to ref. [11] for further details and for references to original publications.

3. SELECTED EXAMPLES

In this section we present typical applications of the quasiclassical method to Josephson weak links involving superconductors with unconventional pairing or the p-wave superfluid \( ^3\text{He} \), which is our paradigm of a system with unconventional pairing. Several aspects of such unconventional Josephson contacts have been discussed by various groups, using either the Ginzburg–Landau theory [19–22] or the tunneling model [23–26]. Recent applications of the quasiclassical theory to weak links in unconventional systems have been published by Rainer and Lee [27], Kopnin [17,28], and Millis et al. [29].

In the following we discuss three selected problems which can readily be solved by the quasiclassical theory, but that are probably too involved to be attacked successfully by other methods. First we discuss the influence of the layers of distorted order parameter near surfaces and interfaces on the Josephson coupling across the link (example A). The second problem deals with the effect of microscopic roughness of the weak link on the critical Josephson current (example B), and finally we use the quasiclassical theory to study the consequences of spin–flip processes on the Josephson–coupling of triplet and singlet superconductors (example C). All three problems are concerned with microscopic supercurrents, and we can profitably use the Matsubara propagators \( \hat{g}(s;\varepsilon;\kappa) \) at imaginary energies \( \varepsilon = i\kappa = i(2\hbar n + 1)T \).

The characteristic curves of the quasiclassical differential equation (1) are straight trajectories of classical high speed particles moving with the velocity \( V(s) \). Of specific importance for the Josephson coupling are classical trajectories connecting the two sides of a junction (see Fig. 1). In fact, for a given order parameter field \( \Delta(s;\kappa;\varepsilon;\kappa) \) on both sides, the Josephson current is completely determined by the connecting trajectories, and is a sum of contributions from the individual trajectories. At interfaces and surfaces the classical trajectories might be deflected or reflected as indicated in Fig. 1b. A calculation of the Josephson current is usually done in three steps. First one has to calculate (self-consistently) the order parameter field. Then one solves the transport-type equation (1) along connecting trajectories, and finally one adds up the contributions from the trajectories to calculate the total Josephson current:

\[
J = T \sum_{\varepsilon} \int ds 2N(s) \Delta \frac{d^2S}{d\varepsilon} L(s;\kappa;\varepsilon;\kappa) g(s;\kappa;\varepsilon;\kappa)
\]

where \( N(s) \) is the local density of states of the normal state at point \( s \) on the Fermi surface, \( d^2S/d\varepsilon \) means integration over the whole Fermi surface, \( d^2S \) denotes the surface integral over a cross section of the link, and \( g \) is the diagonal, scalar part of the quasiclassical propagator in the Matsubara representation.

Example A: Order parameter depletion layer

The anisotropic order parameter of systems with unconventional pairing reacts sensitively to quasiparticle scattering at impurities, surfaces and interfaces. The order parameter will be reduced and distorted within distances from the scattering centers of the order of the coherence length \( \xi(T) \). This leads to new effects which are not found in systems with isotropic pairing. Of importance for our discussion are layers of a depleted order parameter covering the weak link (see Fig. 2). These interlayers between two bulk electrodes (reservoirs) will, in general, lower the Josephson coupling. Kopnin [28] has shown that the Josephson critical current through a small aperture in a diffusive wall is strongly reduced near \( T_c \) by the depletion layers. The Josephson current vanishes proportionally to \((T_c-T)^2\), replacing the usual linear law in \((T_c-T)\). The strong reduction effect disappears at lower temperatures where the thermal coherence length \( \xi_{th} = h\nu_0/kT \) exceeds the thickness \( \xi(T) \) of the order parameter depletion layers, and phase coherence can be sustained across the layers. The recovery
of the critical Josephson current at low T is shown in Fig. 3 which displays numerical results.

Fig. 2. Schematic drawing of the layers of a distorted order parameter (depletion layers) in unconventional systems. The layer is microscopically small on the conventional side of the weak link of Fig. 2(b).

Fig. 3. Temperature dependence of the critical current of an S–N–S weak link, with the S-electrodes representing an axial p-wave state (dashed line) and a d-wave state (solid line) with \( \Delta(\tilde{p}) = \Delta_1 p_x + \Delta_2 p_y p_z \). The critical current is normalized by \( J_0^C \), the critical current in the absence of the depletion layers.

Fig. 4. Order parameter of a d-wave superconductor \( \Delta(p) = \Delta_1 p_x + \Delta_2 p_y p_z \) near a fully transparent S–N interface at \( T/T_C = 0.6 \). The upper and lower curves refer to \( \Delta_1 \) and \( \Delta_2 \) respectively, but we see no major difficulties in using the quasiclassical approach for a numerical analysis of more realistic weak links.

Example B: Microscopic roughness

Quasiparticle scattering in the weak link affects the Josephson coupling by breaking (unconventional) Cooper pairs while they are tunneling across the weak link. We emphasize that this effect is different from the effect of the order parameter depletion layers discussed above, although both effects have the same origin, the scattering of quasiparticles by microscopic imperfections. The depletion layers already exist in the absence of a Josephson current, whereas the present effect requires a finite Josephson current. We will discuss the pairbreaking effect on tunneling Cooper pairs in a model S–N–S junction, where N is a dirty normal layer characterized by its thickness d and mean free path \( \xi \). We consider the limit of a thin dirty layer, i.e., \( \xi \ll d \) and \( \xi < d \). This thin dirty layer describes realistic S–N–S junctions with a dirty normal metal sandwiched in between two superconductors, and also serves as a general model for interface roughness [12,17,31]. The Josephson current of this S–N–S weak link has been calculated for conventional superconductors by Kullik and Omelyanchuk [32]. They found for two identical superconductors the Josephson current

\[
J(\psi) = \frac{1}{\sin \psi} \frac{4\pi k T_N}{\hbar} \frac{\Delta \sin(\psi/2)}{2} \arctg \frac{\Delta \sin(\psi/2)}{\hbar} \tag{6}
\]

Here, \( \Delta^2 = \Delta^2 \cos^2(\psi/2) + \frac{\xi^2}{d} \), \( \psi \) is the phase difference, and \( T_N \sim d/\xi \) is the normal resistance of the dirty layer.

The quasiclassical equations for the thin dirty layer are:

\[
[ \hat{\mathbf{g}}_L (\zeta; \xi) , \hat{\mathbf{g}}_L (s, \zeta; \xi) ] = -\frac{2\pi i}{\hbar} \hat{\mathbf{g}}_L (s, \zeta; \xi) \tag{7}
\]

where the angular brackets denote averaged quantities over the Fermi surface.

\[
\langle \hat{\mathbf{x}} \rangle = \frac{1}{N(E_F)} \int d^2 s N(s) \hat{\mathbf{x}}(s)
\]

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and $\rho$ is the dimensionless parameter $d/\xi$. The propagator $\hat{g}_L$ at $\xi=0$ and $\xi=1$ must match continuously the quasiclassical propagators in the left and right electrodes. In the dirty limit (d/$2\Omega$) $\hat{g}_L$ becomes independent of $s$, and one can replace eq. (7) by the corresponding Usadel's equation:

$$\partial \frac{\hat{c}_{\xi n}^r}{\xi n} - \frac{\partial}{\partial \xi} \hat{c}_{\xi n}^l = 0 \quad (8a)$$

$$\hat{g}_L (\xi, x)^2 = n \quad (8b)$$

which can be solved exactly. The general solution is $\hat{g}_L = \exp(-\alpha \xi)$, for any $\alpha$, satisfying $b^2 = \pi^2$ and $(\alpha, b) = 0$. Equations (8a,b) are a good approximation to eq. (7) in the whole $N$-region except for small boundary layers of size $\epsilon/d$ near $\xi = 0$ and $\xi = 1$. The supercurrent density in the layer is given by the dirty limit formula:

$$j = T_S \Sigma d_s n_s (s' F_{\gamma n}^r n_F \frac{d}{\xi n} \frac{\partial}{\partial \xi} \hat{c}_{\xi n}^r \hat{c}_{\xi n}^l) \quad (9)$$

Matching the solution of eqs. (8a,b) at $\xi=0$ and $\xi=1$ to the bulk propagators, leads, in the case of conventional superconductors, to formula (6) for the Josephson currents. The analogous calculation for systems with unconventional pairing can only be done numerically [30], mainly because the depletion layers preclude an analytic solution of eq. (1). For specific unconventional states, however, one can show by symmetry arguments that the Josephson current is exponentially small. This result is obtained, for example, if the order parameter in both electrodes are odd functions under reflection at a plane or line perpendicular to the $N$-$S$ boundary. Particular examples of importance are the properly oriented axial state of $\text{He}^-$ ($\xi$ normal to the boundary), and the d-wave state $\Delta(\mathbf{p}) \approx (\xi^x \mathbf{p} + \xi^y \mathbf{p}^2)$ for either $\xi^x$, $\xi^y$, or $\xi^z = \alpha x + \beta y$ oriented normal to the boundary. In our planar geometry such states lead to vanishing anomalous parts of the Fermi surface averaged propagator $\hat{g}_L$. This holds, in particular, inside the dirty layer, and one finds using the solution of eqs. (8a,b): $\hat{g}_L \approx \exp(\alpha \xi)$. The propagator $\hat{g}_L$ has no anomalous part. Consequently, this solution carries no supercurrent as can be seen from eq. (9). We expect an exponentially small Josephson current of the order $\exp(-d/\xi)$ coming from corrections to the above, which are not included in Usadel's dirty limit theory. This result should be compared with eq. (6) for conventional superconductors which yields a sizeable Josephson current of the order $d/\xi$. An exponentially strong reduction of the Josephson current by disorder in the weak link requires particular symmetries of the unconventional order parameter. Other unconventional states will also show some reduction of the Josephson current as compared to conventional states, because at least part of the anomalous propagator in the weak link will be washed out by quasiparticle scattering.

Example C: Spin-active barrier

As was first noticed by Fens et al. [23] no Josephson coupling of the usual type exists for a contact of a singlet with a triplet superconductor. This rule is correct as long as the barrier does not flip the electron spins. The possibility of a Josephson coupling induced by spin-active barriers has been discussed in the framework of the Ginzburg-Landau theory by Fenton [19], and Geshkenbein and Larkin [20], and, in a different framework, by Sauls et al. [25] who used a spin-dependent generalization of the tunneling model of Ambegaokar and Baratoff [33]. Both approaches have some limitations. The G-L theory is restricted to temperatures near $T_c$, and the tunneling model cannot handle spatially inhomogeneous situations which are of special interest in unconventional superconductors because of the layer of broken superconductivity at the barrier. A more comprehensive model for a spin-active barrier has been developed by Millis et al. [29] using the quasiclassical approach. The barrier is characterized phenomenologically by a scattering S-matrix for quasiparticles. The four components of the S-matrix $(S_{11}, S_{12}, S_{21}, S_{22})$ relate in the usual way the outgoing quasiparticle states on both sides of the barrier to the incoming ones. Each component is a Nambu matrix in the spin and particle-hole degrees of freedom of quasiparticles. According to Millis et al. a boundary condition at the barrier can be formulated in terms of this S-matrix, which relates the quasiclassical propagators on the two sides (left, right) of the barrier. For a weakly transmitting interface without roughness they obtain the following boundary condition

$$\hat{g}_L (s_{1 n}, \xi_n^r) \times - \frac{1}{2\Omega} [S_{21} (S_{12}^g + \text{in})] \times \hat{g}_L (s_{2 n}, \xi_n^r) \times$$

$$\hat{g}_L (s_{1 n}, \xi_n^r) \times - \frac{1}{2\Omega} [S_{12}^g (S_{21}^g + \text{in})] \times \hat{g}_L (s_{2 n}, \xi_n^r) \times$$

The Fermi surface parameters $s_{1 n}, s_{2 n}, s_{1 n}, s_{2 n}, s_{2 n}$ fix the momenta of incoming (in) and outgoing (out) quasiparticles on the left (l) and right (r) sides of the barrier. The four momenta are determined completely in terms of any one of them by momentum conservation along the barrier (see ref. [29]). Model building of the S-matrix has to take into consideration various conservation laws (particle conservation, etc.) and symmetries of the scattering matrix (spin- and particle-conservation, etc.). This reduces the number of phenomenological parameters describing a barrier in the quasiclassical scheme.

The following procedure is useful for calculating the Josephson current in the case of a weakly transmitting barrier. One first has to determine the quasiclassical propagators at the barrier in the limit of a totally reflecting barrier $(S_{12} = S_{21} = 0)$. This is done by solving eq. (1) in both electrodes subject to the boundary condition of total reflection. The Josephson current is then obtained in leading order in $S_{12}^g = 1/2S_{21}^g$ by inserting the propagators into the right hand side eq. (10a) (or alternatively $(\delta_{0b})$ which gives us the leading correction to $g_1(s_{1 n}) - S_{12}^g (s_{2 n})$). The sum of propagators is sufficient for calculating the current from formula (6). The calculation can readily be done for isotropic superconductors because the $g$ are insensitive to the barrier, and one can simply insert the bulk propagators into the right hand side of eqs. (10a,b). One then finds the well known result: of Ambegaokar and Baratoff [33] for the DC-Josephson current in the tunneling model:}

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\begin{equation}
\frac{n}{4} \langle | S_{21} |^2 \rangle N(E_F) v_{F} \sum_{\ell} \frac{|1^\prime \ell|}{\sqrt{\varepsilon_n^2 + |1^\prime \ell|^2}} \sin(\psi)
\end{equation}

The angular bracket denotes an appropriate average of the transmission probability \(|S_{21}|^2\). The same method applied to a singlet-triplet Josephson contact gives the following result for \(j\)

\begin{equation}
\int d^2 s N(s) v_{F} \sum_{\ell} \frac{2 \operatorname{Re}(C_{21}^\prime C_{11}^\prime) \operatorname{Im}(\Delta_0 \Delta_{\ell}^\prime)}{\sqrt{\varepsilon_n^2 + |1^\prime \ell|^2}} \frac{\sqrt{\varepsilon_n^2 + |1^\prime \ell|^2}}{\sqrt{\varepsilon_n^2 + |1^\prime \ell|^2}}
\end{equation}

Here, \(\Delta_0\) and \(\Delta_{\ell}\) are the singlet and triplet order parameters, \(S_{21}\) and \(C_{11}\) are parameters of the S-matrix describing transmission across the barrier without spin-flip and with spin-flip. Details of the notation are explained in ref. [29]. Equation (11) is derived for a barrier which preserves time reversal invariance. There are no general symmetry reasons for \(j\) to vanish. This confirms the results of Sauls et al. [26] who predicted a Josephson coupling of conventional magnitude for systems with strong spin-orbit scattering. The actual size of the Josephson coupling will depend sensitively on the detailed momentum dependence of the S-matrix parameters \(S_{21}\) and \(C_{11}\). The integrand in eq. (11) has no unique sign, and cancellation effects might reduce the Fermi surface integral to some degree. Equation (11) has been obtained without taking into account the depletion of the order parameter near the barrier or the effects of barrier roughness. The depletion will reduce the coupling near \(T_c\), but should not lead to dramatic effects at lower temperatures. Barrier roughness, on the other hand, might influence strongly the Josephson effect in unconventional superconductors.

4. Summary

A calculation of the coupling strength for Josephson junctions involving systems with unconventional pairing requires the consideration of a variety of effects which are unimportant for conventional junctions, but which have a decisive influence on the properties of "unconventional" junctions. These include pairing breaking by quasiparticle reflection at the barrier, the effect of barrier roughness, and spin-flip processes in the barrier. On a qualitative level each of these effects is quite well understood. A quantitative calculation, however, is a complex problem requiring continued development of powerful computer routines. The quasiclassical formulation of the theory of superconductivity and of superfluid He is the most promising approach for this purpose. Most of the necessary "tools" including suitable equations for equilibrium and non-equilibrium effects as well as boundary conditions at walls, barriers and interfaces are available, and first numerical studies have demonstrated the capacities of quasiclassical methods for handling complex problems in junctions with unconventional systems. The best testing ground for the quasiclassical methods is certainly superfluid He, which shows all the subtleties of unconventional pairing but is not plagued by an abundance of materials parameters. A future observation of Josephson effects in \(^3\)He would probably stimulate further applications of the quasiclassical theory to the study Josephson weak links in systems with unconventional pairing.

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