Acoustic Order Parameter Three-Wave Resonance in Superfluid $^3$He-B.

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Abstract. - We consider the nonlinear interaction of zero sound with the collective modes of the order parameter in superfluid $^3$He-B. Starting from the quasi-classical theory of superfluid $^3$He, we derive equations describing a three-wave resonance between the $J = 2^+$ (real squashing) modes and two zero-sound phonons at different frequencies. We point out similarities between the nonlinear excitation of collective modes of the $^3$He-B order parameter and related effects in nonlinear optical systems. In particular, it should be possible to observe stimulated Raman scattering of zero sound and two-phonon absorption by the $2^+$ mode.

Zero sound has been used extensively to study the collective modes of the order parameter of superfluid $^3$He-B [1]. Although the condensate of Cooper pairs in $^3$He-B is not a simple collection of molecules, many of the observed properties of the absorption and group velocity spectra of zero sound, such as the Zeeman splitting of the collective mode absorption peaks, have analogues in the optical spectroscopy of atoms and molecules. Most experiments have been carried out in the linear response regime in which sound couples only to modes of the same frequency and wavevector. However, it is natural to investigate whether in superfluid $^3$He there exist acoustic analogues of the nonlinear effects which have been studied in optical systems.

Some nonlinear optical effects occur because the intense electromagnetic wave causes a macroscopic population of one or more of the excited states of the medium. Consequently, the population of the ground state of the system must be included as a dynamical variable in the equations describing these phenomena. Examples of this class of effects are population inversion, saturation effects and self-induced transparency [2]. Another class of nonlinear effects are parametric processes such as harmonic generation, stimulated Raman scattering and two-photon absorption [2, 3]. Here the population of the ground state is usually not treated as a dynamical variable, but is assumed to have its equilibrium value. Except for a paper by Serene [4] on third-harmonic generation, previous investigations [5-7] of nonlinear
sound propagation in superfluid $^3$He have focused on effects caused by a macroscopic population of excited states.

Parametric processes involve the absorption and emission of modes with differing frequencies, and are common in nonlinear optics, plasma physics, electronics, acoustics and fluid dynamics [8]. The simplest parametric process is a *three-wave resonance* in which nonlinear interactions allow two linear modes with frequencies $\omega_1$ and $\omega_2$, and wavevectors $\mathbf{q}_1$ and $\mathbf{q}_2$, to excite a third mode with frequency $\omega_3$ and wavevector $\mathbf{q}_3$, given by

$$\omega_3 = \omega_1 + \omega_2, \quad \mathbf{q}_3 = \mathbf{q}_1 + \mathbf{q}_2.$$  \hspace{1cm} (1)

The inverse process, the excitation of modes 1 and 2 by mode 3, is also allowed. In quantum theory such a parametric process is interpreted as the decay of a quantum of mode 3 into quanta of modes 1 and 2, and eqs. (1) express the conservation of energy and momentum for the process.

We focus on two parametric processes that are possible in superfluid $^3$He: i) the excitation of an order parameter collective mode in $^3$He-B, called the real-squashing mode, by two zero-sound phonons, and ii) the decay of a zero-sound phonon into a quantum of the real-squashing mode, which we shall call a real *squashon* hereafter, and a second zero-sound phonon.

Two important questions need to be addressed by a theory of parametric processes in superfluid $^3$He:

1) Is the parametric process allowed by symmetry for the modes involved; i.e. what are the selection rules?

2) What are the energy density requirements for experimental detection of parametric excitation of the modes?

The collective modes of the order parameter in the $B$ phase of superfluid $^3$He can be classified according to the quantum numbers $J, M, \pm$ where $J = \{0, 1, 2\}$ is the total angular momentum, $M = \{-J, \ldots, 0, \ldots, J\}$ is the magnetic quantum number and $+ (-)$ denotes the real (imaginary) part of the order parameter [9]. Serene [lo] has discussed in detail the constraints which the symmetries of liquid $^3$He place on the linear coupling between zero sound and the different order parameter collective modes. An important selection rule is imposed by particle-hole symmetry of the $^3$He Fermi liquid (1). It is useful to formulate this symmetry in terms of a unitary transformation $C$ which maps quasi-particle states just above the Fermi surface into quasi-hole states just below the Fermi surface (and *vice versa*) [10,11]. Particle-hole symmetry determines the selection rules for the coupling of zero sound to the order parameter collective modes because the real (imaginary) components of the order parameter are even (odd) under $C$ and density fluctuations are odd under the transformation $C$ [10].

The $J=2^+$ mode, also known as the real squashing (RSQ) mode, has frequency of approximately $1.1A(T)$, where $A(T)$ is the equilibrium energy gap. The dynamical equations for the RSQ modes in the linear response limit are

$$\ddot{\chi}(\omega)\left[(\omega + i\gamma)^2 - \omega_{M=} (q)^2\right]D^M_{\omega} (\omega, q) = \frac{6}{1 + F_0} \beta_M \tilde{n}(\omega, q),$$ \hspace{1cm} (2)

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(1) Particle-hole symmetry is only an approximate symmetry in $^3$He because the density of states above and below the Fermi surface are slightly different.
where \( D_M^\nu(\omega, q) \) is the amplitude of the mode with magnetic quantum number \( M \), \( 1/\Gamma \) is the lifetime of the mode due to quasi-particle collisions, \( \tilde{\lambda}(\omega) \) is the Tsuneto function and \( \delta n(\omega, q) \) is the density fluctuation. These modes have the opposite signature from zero sound under the particle-hole symmetry transformation, and so couple only weakly to sound as a result of the small-particle hole asymmetry in \(^3\)He (see \(^1\)). Consequently, the coupling constant \( \beta_M \) is small, of order \( r = N'(E_F) \Delta/N(E_F) \approx 10^{-5} \) \([12]\).

The propagation of sound in superfluid \(^3\)He is described by a wave equation

\[
\left[ \frac{\partial^2}{\partial t^2} - c_1^2 \nabla^2 \right] \delta n = 2c_1^2 \nabla^2 \delta \Pi, \tag{3}
\]

where \( \delta \Pi(\mathbf{R}, t) \) is related to the stress tensor, and \( c_1 \) is the hydrodynamic sound velocity. Equation (3) is a consequence of the mass and momentum conservation laws. It is important to note that although this equation is linear in \( \delta n \) and \( \delta \Pi \) it describes the nonlinear sound propagation because the longitudinal stress \( \delta \Pi \) is in general a nonlinear functional of the density fluctuation and, in general, the amplitudes of the collective modes of the system which couple to zero sound. The relationship between the fluctuating stress \( \delta \Pi \) and the density fluctuation \( \delta n \) must be obtained from a microscopic theory \(^2\). The frequency-dependent attenuation \( \alpha \) and phase velocity \( c(\omega) \) of sound are related to the stress \( \delta \Pi \) by

\[
\alpha = -q \text{ Im} \left\{ \frac{\delta \Pi}{\delta n} \right\}, \quad \frac{c(\omega) - c_1}{c_1} = \text{ Re} \left\{ \frac{\delta \Pi}{\delta n} \right\}. \tag{4}
\]

Although the RSQ modes can only be excited by a single zero-sound phonon via the weak particle-hole asymmetry in \(^3\)He, the excitation of the RSQ mode by two-phonon processes is not forbidden by particle-hole symmetry selection rules \(^3\). Thus, at higher sound amplitudes the right-hand side of eq. (2) contains a driving term which is second order in the density; and the stress tensor has a term which is bilinear in \( \delta n \) and the RSQ mode amplitude \( D_M^\nu \). We use the quasi-classical theory \([13]\) of superfluid \(^3\)He to derive these nonlinear couplings, which are allowed on symmetry grounds \([14]\). In particular, we obtain

\[
\delta \Pi(\omega) = \frac{1}{(1 + F_0^\nu)\Delta} \sum_M \int d\nu A^\nu(\omega, \nu) \delta n(\nu) D_M^\nu(\omega, \nu), \tag{5}
\]

\[
\tilde{\lambda}(\omega) [\omega + i\Gamma]^2 - \omega_M^2 (q)^2] D_M^\nu(\omega, q) = \frac{6}{(1 + F_0^\nu)\Delta} \int d\nu A^\nu(\nu) \delta n(\nu) \delta n(\omega - \nu), \tag{6}
\]

where \( A^\nu \) is a dimensionless function of order one which is discussed in more detail later. These are the central equations describing parametric excitation of the RSQ modes by sound. The detailed derivation of the equations from microscopic theory is given elsewhere \([14]\). However, it is worth noting that the coupling function \( A^\nu \) may be obtained

\(^3\) This is analogous to the situation in optics, where the propagation of electromagnetic waves is described by the wave equation

\[
\left( \frac{\partial^2}{\partial t^2} - c_e^2 \frac{\partial^2}{\partial x^2} \right) \mathbf{E} = -4\pi \frac{\partial^2}{\partial t^2} \mathbf{P}.
\]

The polarization \( \mathbf{P} \) is related to the electric field \( \mathbf{E} \) by a constitutive relation which can be nonlinear.

\(^3\) These selection rules are similar to those for optical processes in media with inversion symmetry: one- (two) photon processes between states of the same parity are forbidden (allowed).
from a diagrammatic perturbation theory which shows that the same coupling function $A^M$ must appear in both the nonlinear stress (5) and the nonlinear driving term for the $J = 2^+$ mode (6) and that the coupling function satisfies the identity $A^M(\omega, \nu, \omega - \nu) = A^M(\nu, -\omega, \omega - \nu)$. These facts are analogous to the permutation identity satisfied by nonlinear optical susceptibilities, and are related to the Manley-Rowe relations of nonlinear optics [3] which state that for the parametric process of eqs. (1), the rate of change of the number density of quanta of mode 3 must equal the rate of change of the number density of quanta of mode 1 (and similarly for mode 2).

We now apply eqs. (5) and (6) to the nonlinear interaction of two zero-sound waves. We separate the density

$$\partial n(R, t) = \text{Re} [N_1(R, t) + N_2(R, t)],$$

where $N_j(R, t) = \tilde{N}_j(R, t) \exp[i(\omega_j t - \mathbf{q}_j \cdot \mathbf{R})], j = 1, 2$ and the wave amplitudes $\tilde{N}_j(R, t)$ vary slowly on the time scale of the mode lifetime $1/\Gamma$. In this quasi-steady-state approximation eq. (5) can be solved for $D_2^R(R, t)$. It is straightforward to show that the RSQ mode contains terms oscillating with frequencies $0, 2\omega_1, 2\omega_2, \omega_1 + \omega_2,$ and $\omega_1 - \omega_2$. If these solutions, together with eq. (6), are substituted in eq. (5) it is found that $D_2^R(R, t)$ contains terms oscillating with frequencies $\omega_1, \omega_2, 3\omega_1, 3\omega_2, 2\omega_1 \pm \omega_2, 2\omega_2 \pm \omega_1$.

The term with frequency $\omega_1$ is given by

$$\frac{\partial \Pi_1(R, t)}{N_1} = [\chi^{(3)}(\omega_1, -\omega_2, \omega_1 + \omega_2) + \chi^{(3)}(\omega_1, \omega_2, \omega_1 - \omega_2)]|N_2|^2,$$

where the nonlinear susceptibility is

$$\chi^{(3)}(\omega, \nu, \omega - \nu) = \frac{6}{5d^2(1 + F)^3} \sum_{M} \frac{|A^M(\omega, \nu, \omega - \nu)|^2}{\lambda(\omega - \nu)((\omega - \nu + i\Gamma)^2 - \omega^2)}.$$

If the wave with frequency $\omega_2$ is of much higher intensity than the wave with frequency $\omega_1$, then the intensity $|N_2|^2$ can be assumed to be a constant. The attenuation and shift in phase velocity of the sound wave with nominal frequency $\omega_1$, due to the nonlinear interaction with the density disturbance at frequency $\omega_2$ and the collective modes of the medium, is calculated from eqs. (4), (8) and (9). We find well-defined features at the resonance conditions $\omega_1 \pm \omega_2 = \omega_{M+}$, corresponding to two-phonon absorption (+) and stimulated Raman scattering (−) of phonons by the RSQ modes.

The coupling constant $A^M$ can be written in the form

$$A^M(\omega, \nu, \omega - \nu) = Z^M(\hat{q}_1, \hat{q}_2) \tilde{A}(\omega, \nu, \omega - \nu),$$

where $Z^M(\hat{q}_1, \hat{q}_2)$ depends only on the magnetic quantum number $M$ and the direction of the two sound wavevectors, such that

$$|Z^M|^2 = \frac{8\pi}{15} |Y_{2,M}(\theta)|^2,$$

where $Y_{2,M}$ is the $J = 2, M$ spherical harmonic. If either $\hat{q}_1$ or $\hat{q}_2$ is parallel to the quantization axis of the modes then $\cos(\theta) = \hat{q}_1 \cdot \hat{q}_2$. If $\hat{q}_1$ and $\hat{q}_2$ are parallel, then $\theta$ is the angle of propagation relative to the magnetic-field $\cos(\theta) = \hat{q} \cdot \hat{z}$, where $\hat{z}$ is the direction of the magnetic field.

The change in the phase velocity $\Delta c$ and the attenuation $\alpha$ near one of the nonlinear
resonances is of the order

\[ \frac{\alpha}{q} c = \frac{\Delta c}{c} \approx \frac{|\hat{A}|^2}{(1 + F_0^2)^2} \frac{\Delta U}{U_c}, \]  

(12)

where \( U \) is the energy density in the high-intensity sound wave and \( U_c = \frac{1}{2} N(E_F) \Delta^2 \) is the superfluid condensation energy density. It is clear from eq. (12) that the nonlinear features will be largest at low pressures (where \( F_0^2 \) is smallest) and low temperatures where \( \Gamma \) is small. Figures 1a) and b) show the predicted temperature dependence of the attenuation

![Graph](image)

Fig. 1. – a) The predicted temperature dependence at zero pressure of the attenuation (in units of cm\(^{-1}\)) of a 38.3 MHz zero-sound wave in the presence of a parallel 3.45 MHz wave of energy density \( 0.2U_c \). b) The attenuation and amplification of the 3.45 MHz wave when the 38.3 MHz wave has energy density \( 0.2U_c \). In both figures the features at \( T/T_c \sim 0.43 \) and 0.72 are due to nonlinear effects.

(or amplification) of a sound wave of frequency \( \omega_1 \) in the presence of a second wave of high intensity and frequency \( \omega_2 \). The large central feature in fig. 1a), occurring at a temperature \( T_0 \) such that \( \sqrt{8/5} \Delta(T_0) = \omega_1 \), is due to the linear coupling of the sound to the RSQ mode as a result of particle-hole asymmetry. The features on the left are due to two-phonon absorption and occur at a temperature \( T_+ \) such that \( \omega_1 + \omega_2 = \sqrt{8/5} \Delta(T_+) \). The features on the right are due to stimulated Raman scattering and occur at a temperature \( T_- \) such that \( |\omega_1 - \omega_2| = \sqrt{8/5} \Delta(T_-) \). The 3.45 MHz sound wave is amplified near \( T_- \) because the 38.3 MHz phonons decay into real squashons and 3.45 MHz phonons. In an actual experiment, because of heating effects, it may be desirable to use a smaller sound energy density than the value...
$U/U_c = 0.2$ used in the calculation for fig. 1. For comparison the estimated energy density in the nonlinear experiments described in ref. [5] is $U/U_c = 0.01$; such a smaller value of $U/U_c$ substantially reduces the size of the nonlinear features, however, they should still be observable because changes in the attenuation of order $0.1\, \text{cm}^{-1}$ and changes in the phase velocity of order $10^{-8}$ are detectable.

In the above discussion we neglected all terms in the stress tensor except those oscillating with the frequencies $\omega_1$ and $\omega_2$. These other terms correspond to sound waves such as the third harmonic at frequency $3\omega_1$ and the anti-Stokes waves with frequency $(2\omega_1 - \omega_2)$. However, because of dispersion a wave with frequency $3\omega_1$ has a wavevector $k_3 \neq 3k_1$, and it can be shown that the wave is destroyed by interference in a distance of order $1/\delta k$, where $\delta k = k_3 - 3k_1$. In $^3$He-B the dominant contribution to $\delta k$ comes from the dispersion due to the $J = 2$-squashing mode. Typically, $1/\delta k \sim 1\, \text{mm}$ and thus the sound cell must be smaller than a millimeter in order to observe the third harmonic. Similar arguments apply to the anti-Stokes waves. It can also be shown that both waves are negligible compared to the waves with frequency $\omega_1$ and $\omega_2$ and so neglecting them earlier was justified. Nevertheless, third-harmonic generation and anti-Stokes waves should still be detectable provided that $2\omega_1 = \omega_{M^+}$ so that the nonlinear susceptibility is large [14].

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