Nonlinear acoustic effects in superfluid $^3$He-B

Ross H. McKenzie and J.A. Sauls

$^a$Department of Physics, Ohio State University, Columbus, OH 43210, USA
$^b$Department of Physics and Astronomy, Northwestern University, Evanston, IL 60208, USA

We consider the nonlinear interaction of zero sound with the collective modes of the order-parameter in superfluid $^3$He-B. The approximate particle-hole symmetry of the $^3$He-Fermi liquid determines selection rules for the linear and nonlinear coupling of zero sound to the collective modes. Starting from the quasiclassical theory of superfluid $^3$He, we have shown that the coupling strengths have a simple representation in terms of Feynman diagrams. We predict measurable two-phonon absorption and nonlinear-Raman scattering by the $J = 2^+$ (real squashing) modes at low pressures. Recent observations of two-phonon absorption by a group in Helsinki are compared to the theoretical predictions. Two-phonon absorption can be used to determine the dispersion of the $J = 2^+$ modes.

1. Introduction

In a conventional superconductor, the Cooper pairs are in an s-wave spin singlet state. The order parameter has two collective modes, the amplitude mode and the phase mode, and these are of little practical interest as far as superconducting properties are concerned. In contrast, the superfluid phases of $^3$He have a rich spectrum of order-parameter collective modes associated with the p-wave spin triplet state of the Cooper pairs. Ultrasound, which alone couples to many of these modes, has proven to be a very useful probe of the superfluid phases. Most experiments have been performed in the linear response regime. In the isotropic B phase there are similarities between the ultrasound spectroscopy of order-parameter collective modes and the optical spectroscopy of atoms and molecules. This has motivated us to investigate acoustic analogues of nonlinear optical effects such as two-photon absorption, stimulated Raman scattering and third-harmonic generation.

In the next section, we give a brief review of the aspects of collective modes and zero sound needed to understand the rest of the paper. More details can be found in the comprehensive review of Halperin and Varoquaux [1]. We show how the approximate particle-hole symmetry of the $^3$He-Fermi liquid determines important selection rules for the linear and nonlinear coupling of zero sound to the order-parameter modes. In section 3 a brief outline is given of how the nonlinear acoustic response is calculated from quasiclassical theory. The nonlinear interaction of the $J = 2^+$ modes with two zero sound waves is discussed in section 4. We predict measurable two-phonon absorption and stimulated Raman scattering by the $J = 2^+$ modes. In section 5, recent observations of two-phonon absorption are compared to our predictions. Finally, other possible nonlinear experiments are discussed.

2. Ultrasound spectroscopy of order parameter collective modes

2.1. Symmetry and quantum numbers

In the superfluid phases of $^3$He, quasiparticles form Cooper pairs in a p-wave ($L = 1$) spin-triplet ($S = 1$) state. The superfluid order parameter is the Cooper pair wave function which can be written in terms of a $3 \times 3$ complex matrix $A_{\mu \nu}$, where $i$ and $\mu$ are spin and orbital indices, respectively. Hence, there are 18 degrees of freedom associated with this order parameter.

In the isotropic B phase, the equilibrium order
parameter can be written in the form

$$A_{\mu} = \Delta(T) e^{i\Phi} R(\hat{n}, \theta)_{\mu},$$  \hspace{1cm} (1)

where $\Delta(T)$ is the temperature dependent BCS energy gap, $\Phi$ is a phase, and $R(\hat{n}, \theta)$ is an orthogonal matrix representing a relative rotation of the spin and orbital coordinates by an angle $\theta$ about the direction $\hat{n}$. If the nuclear dipole energy is neglected then any order parameter of the form (1) minimizes the free energy of the superfluid. The nuclear dipole energy forces $\theta = 104^\circ$ [2]. Although this partially lifts the degeneracy, the order parameter (1) is still invariant under rotations generated by the operator

$$J = L + R^{-1}(\hat{n}, \theta) \cdot S,$$

and so the equilibrium state can be assigned total angular momentum $J = 0$. In the B phase, gauge and relative spin–orbital rotational symmetries are spontaneously broken. This is in contrast to a conventional superconductor, in which only gauge symmetry is broken. It is this high degree of symmetry breaking which gives superfluid $^3$He such a rich spectrum of phenomena.

The order-parameter collective modes are oscillations of the order-parameter about the equilibrium value (1). At zero wave vector they can be classified [3] by the quantum numbers $J^2$ and $M_J$, where $J = 0, 1, 2$ is the total angular momentum (corresponding to the three possible pairings of $L = 1$ and $S = 1$), $M_J = \{-J, \ldots, 0, \ldots, J\}$ is the magnetic quantum number and $\xi = \{+, -\}$ is the “signature” under particle–hole symmetry (see the end of this section) corresponding to the real and imaginary parts of the order-parameter, respectively. Note that there are a total of eighteen order-parameter collective modes. The $J = 2^+$ and $J = 2^-$ modes, which are also known as the real and imaginary squashing modes, respectively, are of particular interest because they couple to ultrasound. These modes lie below the pair breaking edge $2\Delta(T)$ (see fig. 1) and are weakly damped by quasiparticle collisions.

The dispersion relations for the collective modes have the form

$$\omega_n(q)^2 = a_n \Delta(T)^2 + b_n(qV_F)^2,$$  \hspace{1cm} (2)

where the subscript $n$ denotes the quantum numbers of the mode, $v_F$ is the Fermi velocity and the coefficients $a_n$ and $b_n$ are weakly temperature dependent functions of order one. Numerous theoretical calculations of these coefficients have been performed. They all work in the mean-field approximation (i.e. neglect strong coupling effects) but vary in how many mean fields are taken into account. The simplest expressions result when the $l = 3$ pairing interactions $v_3$ and the $l = 2$ Fermi liquid parameters $F_2^1$ and $F_2^2$ are set equal to zero. In this case, $a_+ = \frac{5}{8}$ and $a_- = \frac{1}{8}$ for the $J = 2^+$ and $J = 2^-$ modes, respectively. Expressions for $a_+(T)$ and $a_-(T)$ when $v_3$, $F_2^1$ and $F_2^2$ are nonzero have been given by Sauls and Serene [4]. The dispersion coefficients $b_n$ and the effect of a magnetic field on $a_n$ will be discussed below.

### 2.2. Coupling between sound and collective modes

The propagation of longitudinal density oscillations in both the normal and superfluid phases
is described by the wave equation

$$\left( \omega^2 - c_1^2 q^2 \right) \delta n(\omega, q) = 2c_1^2 q^2 \delta \Pi(\omega, q),$$  \hspace{1cm} (3)

where \( \delta n(\omega, q) \) is the density oscillation, \( \delta \Pi(\omega, q) \) is the longitudinal component of the energy momentum stress tensor, and \( c_1 \) is the hydrodynamic sound velocity given by

$$c_1^2 = \frac{v_F^2}{3} \left( 1 + F_0^2 \right) \left( 1 + \frac{1}{2} F_1^2 \right),$$  \hspace{1cm} (4)

where \( F_0 \) and \( F_1 \) are Fermi liquid parameters. The ratio \( v_F/c_1 \) varies from about 0.3 at low pressures to about 0.07 near the melting curve. Consequently, \( (qv_F/\omega)^2 \) is a useful expansion parameter in calculations. In the normal phase, density waves with frequency much larger than \( \omega > 1/\tau \) propagate as a collective mode known as zero sound. This mode has velocity \( c_0 \), a few percent larger than \( c_1 \). This mode also exists in the superfluid phase and couples to the phase of the order-parameter, propagating as the \( J=0^- \) (Goldstone) mode. Its dispersion relation is shown in fig. 1. Note that it crosses the dispersion relations of the \( J=2^+ \) and \( J=2^- \) modes.

The coupling between zero sound and the order-parameter collective modes can be calculated from the quasiclassical theory (see section 3). The collective mode contribution to the stress tensor is, in the linear response limit, of the form

$$\delta \Pi(\omega, q) = \frac{N(E_F)}{1 + F_0^2} \sum_n \beta_n(\omega) D_n(\omega, q),$$  \hspace{1cm} (5)

where \( D_n(\omega, q) \) is the amplitude of the mode with quantum numbers \( n \), \( \beta_n(\omega) \) the corresponding coupling strength, \( N(E_F) \) is the single spin density of states at the Fermi surface, and \( F_0 \) is a Fermi liquid parameter.

The dynamical equations for the modes derived from the quasiclassical theory are of the form

$$\lambda(\omega)(\omega^2 + 2i\omega \Gamma - \omega_n^2(q)) D_n(\omega, q)$$

$$= \frac{\Delta^2 \beta_n(\omega)^*}{N(E_F)} \delta n(\omega, q),$$ \hspace{1cm} (6)

where \( \Gamma \) is the mode damping due to quasiparticle collisions and \( \lambda(\omega) \) is the Tsuneto function given by

$$\lambda(\omega, T) = \Delta^2(T) \int_{\Delta(T)}^\infty \frac{d\epsilon}{\sqrt{\epsilon^2 - \Delta^2(T)}} \tanh(\epsilon/2T).$$ \hspace{1cm} (7)

It follows from the wave equation (3) that a wave of frequency \( \omega \) had attenuation \( \alpha(\omega) \) and phase velocity \( c(\omega) \) given by

$$\alpha(\omega) = -q \text{Im} \left( \frac{\delta \Pi}{\delta n} \right),$$ \hspace{1cm} (8)

$$c(\omega) = c_1 + c_1 \text{Re} \left( \frac{\delta \Pi}{\delta n} \right).$$

The contribution to the attenuation from the collective modes can be found by combining equations (5) and (6). The result is

$$\alpha(\omega) = \frac{12 \Delta^2 q}{(1 + F_0^2)\lambda(\omega)}$$

$$\times \sum_n \frac{\left| \beta_n \right|^2 \omega \Gamma}{(\omega_n^2(q) + 2\omega \Gamma)^2}. \hspace{1cm} (9)$$

There will be features in the attenuation and phase velocity of sound wherever its frequency and wave vector are equal to that of one of the modes to which it couples. In typical ultrasound experiments a sound transducer of fixed frequency is used and the temperature is varied. Consequently, the collective mode frequencies are also varied, and there are distinct temperatures at which resonances of the collective modes occur. Parenthetically, we note that the derivation of (7) and (8) implicitly assumes that \( \alpha \ll q \). This is a good approximation for the \( J=2^+ \) modes but not for the \( J=2^- \) modes at low temperatures. Varoquaux et al. [5] have given a complete discussion of this point.

The calculated temperature dependence of the attenuation of a 36 MHz sound wave at zero pressure is shown in fig. 2. The three distinctive features occurring at temperatures \( T_R \), \( T_1 \), and \( T_{ph} \), are due to the \( J=2^+ \) (real squashing) mode, the \( J=2^- \) (imaginary squashing) mode and pair...
The calculated temperature dependence of the attenuation of a 36 MHz sound wave in superfluid $^3$He-B at a pressure of 0 bar. As the temperature decreases, the energy gap increases monotonically. Consequently, at certain specific temperatures the sound frequency equals the frequency of one of the order parameter collective modes. The large attenuation for temperatures between $T_{pb}$ and $T_{~}$ is due to the breaking of the Cooper pairs by the sound wave. The features at temperatures $T_n$ and $T_c$ are due to coupling of sound to the $J=2^+$ and $J=2^-$ order parameter collective modes, respectively. Note that the attenuation due to the $J=2^+$ mode is several orders of magnitude larger than that due to the $J=2^-$ mode. As explained in the text, this is a consequence of the near exact particle-hole symmetry of the $^3$He Fermi liquid.

The peak in the attenuation due to mode $n$ has height

$$\alpha_{max} = \frac{6A^2|\beta_n|^2}{(1 + F^2)\lambda(\omega)c_0\Gamma}$$

(10)

and temperature width $\delta T$ given by

$$\delta T = \frac{h\Gamma}{k\alpha_n\Delta'(T/T_c)}$$

(11)

where $\Delta'(T/T_c)$ is the magnitude of the slope of the gap function. At low temperatures, the mode lifetime ($1/\Gamma$) becomes very large because the density of quasiparticles is exponentially small due to the finite energy gap ($1/\Gamma \sim \exp(\Delta(T)/T)$ as $T \rightarrow 0$). A group at Northwestern measured the temperature dependence of $\Gamma$ from the off-resonance lineshape of the $J=2^-$ mode [7]. They found that $\Gamma$ decreased by three orders of magnitude from $T_c$ to $0.4T_c$. Consequently, the height of the attenuation peaks should increase as the collective mode resonances occur at lower temperatures. It will be seen in section 4 that this property is important for the nonlinear resonances discussed there. Note that the observed line width can be larger than that predicted by eq. (11) due to inhomogeneous broadening (e.g., a temperature gradient or texture in the sound cell can cause a distribution of mode frequencies).

### 2.3. Effect of a magnetic field

The tensor character of the order-parameter collective modes becomes evident in a magnetic field. A five-fold Zeeman splitting of the $J=2^+$ modes is observed in a large magnetic field [6, 8]. We now discuss how the mode dispersion relations and the size of the coupling of sound to the different modes is affected by a magnetic field. In a magnetic field $H$, the Larmor frequency $\omega_L(T, H)$ is given by

$$\omega_L(T, H) = \frac{gH}{F_0} \frac{1}{3} (2 + Y(T))$$

where $Y(T)$ is the Yosida function. Dynamical equations of the form (6) are actually derived from time-dependent gap equations (see section 3). The symmetric traceless part of the order parameter, $d_{\mu}(\omega, q)$ satisfies an equation of the form

$$(\omega^2 + 2i\omega\Gamma - a\Delta(T)^2 - c_0^2q^2)d_{\mu}$$

$$- c_0^2(q_\mu q_k d_{\mu k} + q_\mu q_k d_{\mu k} - \frac{1}{2}\delta_{\mu k}d_{\mu k}q_kq_\mu)$$

$$+ i\omega\epsilon_{\alpha k l}g\tilde{H}_{\alpha l}d_{\mu k} = \frac{6A^2\beta(\omega)}{\lambda(\omega)} \delta n(q_\mu q_\nu - \frac{1}{2}\delta_{\mu \nu}),$$

(12)
where $\vec{H} = \omega_1(T, H)R(\vec{n}, \theta)\mu \vec{H}_\mu$ is the "effective" magnetic field. (N.B. this may be in a different direction to the actual field). The velocities $c_a$ and $c_b$ are of order $v_F$.

To derive equations of the form (6) this Hermitian matrix equation must be diagonalized. Let \( \{u_{\mu}^m\} \) be the set of five solutions, orthonormalized so that

\[
u_{\mu}^m \nu_{\mu}^m = \delta_{mn}.
\]

If we set

\[
d_{\mu}(\omega, q) = \sum_{n} D_n(\omega, q)u_{\mu}^n,
\]

then the amplitudes $D_n(\omega, q)$ of the different modes must satisfy equations of the form (6) with the coupling strength $\beta_n$ given by

\[
\beta_n(\omega) = \beta(\omega)u_{n}^\mu \hat{\mu} \hat{\mu}.
\]

Simple analytic expressions for $\omega_n(q)$ and $u_{\mu}^n$ are only possible for zero field and magnetic fields at which the Zeeman energy is much larger than the dispersion energy (i.e. $\omega g \gamma H_c^\alpha \gg c_b^2 q^2$). Using the fact that $q \sim \omega/c_1$, and $c_a \sim v_F$, we define a particular magnetic field $H_0$ by

\[
\omega_1(T, H_0) = \omega \left( \frac{v_F}{c_1} \right)^2.
\]

It is useful to introduce the set of orthonormal $J = 2$ tensors:

\[
\begin{align*}
i^0_\mu &= \sqrt{3/2} (\hat{x}_\mu - \frac{1}{2} \delta_{\mu 0}) , \\
i^{\pm 1}_\mu &= \mp (1/\sqrt{2}) \left[ (\hat{x}_\mu \pm i\hat{y}_\mu) \hat{z}_\mu \pm \hat{z}_\mu (\hat{x}_\mu \pm i\hat{y}_\mu) \right] , \\
i^{\pm 2}_\mu &= \mp \frac{1}{2} (\hat{x}_\mu \pm i\hat{y}_\mu, \hat{z}_\mu \pm \hat{z}_\mu) \hat{z}_\mu.
\end{align*}
\]

In zero field, the eigentensors $u_{\mu}^n$ of eq. (12) are the tensors $i^{\alpha}_\mu$, given by (16) with the mode quantization axis $\hat{z} = \hat{a}$ and dispersion relation at $H = 0$

\[
\omega_m(q)^2 = a \Delta(T)^2 + c_b^2 q^2 + \frac{1}{2} \left( 4 - m^2 \right) c_b^2 q^2.
\]

According to (14) only the $m = 0$ mode couples to sound at $H = 0$. The quantum number $m$ defines the projection of the eigentensor along the sound propagation direction.

In a large magnetic field ($H \gg H_0$) the eigentensors $u_{\mu}^n$ go over to the tensors $i^{\alpha}_\mu$ given by (16) with the mode quantization axis $\hat{z} \parallel$ parallel to the effective field $\vec{H}$. The mode dispersion relations are

\[
\omega_m^2(q) = a \Delta(T)^2 + 2 g(T) M \omega_c + c_b^2 q^2,
\]

where $g(T)$ is the Landé factor for the modes and the mode velocities $c_M$ are given by

\[
c_M^2 = c_a^2 + c_b^2 \left[ \frac{1}{2} (1 + \frac{1}{2} M^2) + \cos^2 \phi (1 - \frac{1}{2} M^2) \right],
\]

where $\phi$ is the angle between the direction of sound propagation and the effective field. Hence, there is a five-fold Zeeman splitting of the modes. It follows from (14) that the height of the attenuation peak due to the mode with magnetic quantum number $M$ is proportional to a $J = 2$ spherical harmonic

\[
|\beta_m(\omega)|^2 = \frac{8 \pi}{15} \beta(\omega) Y_{2M}(\phi)^2.
\]

This angular dependence was observed by Avvenel et al. [8].

In the presence of a texture, the angle $\phi$ and consequently the mode frequencies will vary in the sound cell. This can lead to a textural splitting of modes with the same $M$ value. Shivaram et al. [6] observed such an effect in a large magnetic field perpendicular to the sound propagation direction. The frequency splitting $\Delta \omega$ is given by

\[
\Delta \omega = \frac{c_b q^2}{4 \omega} = \frac{\omega}{4} \left( \frac{c_b}{c_0} \right)^2.
\]

We wish to stress that the relation (20) is not valid at all field strengths. For fields smaller than or comparable to $H_0$, eq. (12) must be diagonalized. The mode eigentensors and consequently the mode dispersion relations and coupling
strengths depend on the ratio \((H/H_0)\) and the angle \(\phi\). Fishman and Sauls [9] showed that for \(H \ll H_0\) the mode frequencies were a nonlinear function of field, the linear Zeeman splitting only occurring for \(H \gg H_0\). The \(m = 0\) mode at \(H = 0\) evolves into the \(M = +2\) mode at large fields. It is the highest frequency mode for all fields. The \(m = \pm 1\) modes evolve to the \(M = 0, +1\) modes and the \(m = \pm 2\) modes evolve into the \(M = -1, -2\) modes. Salomaa and Volovik [10] found the collective mode frequencies and coupling strengths as a function of \(H/H_0\) and \(\phi\) by solving a phenomenological equation similar to (12). They found that the relation (20) was valid only for \(H \gg H_0\) (see fig. 2 in ref. 10). For \(H \ll \frac{1}{2}H_0\) the \(M = +2\) \((m = 0)\) mode has the largest coupling to sound. The \(M = -2\) mode has extremely small coupling in this field regime. Except in the high field regime, the modes with quantum numbers \(M\) and \(-M\) have quite different coupling strengths. This asymmetry was observed by Shivaram et al. [6], who found that at low pressures the \(M = +2\) mode was first observable at much lower fields than the \(M = -2\) mode for magnetic fields both parallel and perpendicular to the sound wave propagation direction. They noted that their observations were inconsistent with (20) but did not provide an explanation.

It is important to note that the magnitude of the field \(H_0\) varies significantly with pressure, decreasing from about 50 mT at 0 bar to about 4 mT at the melting curve (34 bar). This will be important in the discussion in section 5 of the observations of two phonon absorption in a magnetic field at low pressures.

2.4. Transient effects

In the derivation of the expression (9) for the sound attenuation, we have implicitly made an approximation for the mode dynamics; i.e. we have assumed the continuous wave (CW) limit which holds if the duration of the sound pulses \(\tau_{\text{pulse}}\) are much longer than the mode lifetime \((\tau_{\text{pulse}} \gg 1/\Gamma)\). In typical experiments the sound pulses are of the order of 10 \(\mu\)s. The mode lifetime becomes of this order for temperatures less than about 0.6\(T_c\) [7]. Avenel et al. [5, 11] found that when the pulse duration was shorter than the mode lifetime, the observed attenuation peak was much smaller and broader than that observed in the CW-limit. Moreover, the area under the peak was smaller. Mast et al. [12] (see fig. 2(b)) observed a similar effect. They measured the height and width of the peak in the attenuation due to the \(J = 2^+\) mode and found that for 4 \(\mu\)s pulses at temperatures less than 0.6\(T_c\), the peak widths and heights were not consistent with an equation of the form (10). Hence, in order to make a quantitative determination of coupling strengths, pulse lengths of sufficient duration must be used.

2.5. Particle-hole symmetry

Important selection rules for the coupling of zero sound to the order parameter collective modes are determined by the approximate particle-hole symmetry of the \(^3\)He Fermi liquid Serene [13] introduced a particle-hole symmetry transformation \(C\) which exchanges a quasiparticle of energy \(\xi_p\) above the Fermi surface with a quasihole of energy \(-\xi_p\), and rotates its spin by \(\pi\). Under this transformation \(C\), a density fluctuation \(\delta n\) is mapped to \(-\delta n\), and an order parameter fluctuation \(\delta A_{\mu}\) is mapped into its complex conjugate \(\delta A_{\mu}^*\). Consequently, we can assign a negative (−) signature (or “parity”) to zero sound phonons and imaginary squashons (quanta of the \(J = 2^-\) mode) and a positive (+) signature to real squashons (\(J = 2^+\)). Hence, if we consider the two processes,

phonon (−) → imaginary squashon (−),
phonon (−) → real squashon (+),

the first is allowed and the second forbidden by particle-hole symmetry. However, Koch and Wölffe [14] showed that there is actually a weak coupling between the \(J = 2^-\) mode and zero-sound because the particle-hole symmetry is weakly broken (the density of states just above and just below the Fermi surface differ slightly). The relative strengths of the coupling of the \(J = 2^+\) mode and the \(J = 2^-\) mode to sound is determined by the particle-hole symmetry factor.
\[ \eta = N'/(E_F^2/N(E_F)) \sim 10^{-3}, \] where \(N(E_F)\) and \(N'(E_F)\) are the density of states and its slope at the Fermi surface, respectively. In particular, the coupling functions \(\beta(\omega)\) in eqs. (5), (6) and (14) are given by

\[ \beta(\omega) = \frac{\omega\lambda(\omega)}{5\Delta}, \]
\[ \beta'(\omega) = \sqrt{2n}\beta_0(\omega), \]

for the \(J = 2^-\) and \(J = 2^+\) modes, respectively.

We now consider what nonlinear processes are allowed by particle-hole symmetry. The process

\[ \text{phonon}(--) + \text{phonon}(-) \rightarrow \text{real squashon}(+) \]

is allowed because the net signature of the left and right hand sides are the same. This process will occur provided that the sound wave frequencies and momenta satisfy

\[ \omega_1 + \omega_2 = \omega_{\text{rsq}}, \quad q_1 + q_2 = q_{\text{rsq}}, \] (22)

and corresponds to two-phonon absorption by the \(J = 2^+\) mode. A similar process is

\[ \text{phonon}(--) \rightarrow \text{real squashon}(+) + \text{phonon}(--), \]

which will occur provided that

\[ \omega_1 = \omega_{\text{rsq}} + \omega_2, \quad q_1 = q_{\text{rsq}} + q_2. \] (23)

This corresponds to Raman scattering of phonons by the \(J = 2^+\) mode. The above two processes are examples of three-wave resonances and the resonance conditions (22) and (23) ensure the conservation of energy and momenta. Clearly, similar processes for the \(J = 2^-\) mode are forbidden by particle-hole symmetry. By similar arguments, it can be shown that second (third) harmonic generation of sound is forbidden (allowed) by exact particle-hole symmetry.

3. Quasiclassical theory of nonlinear acoustic response

Given that certain nonlinear processes are not forbidden by particle-hole symmetry, it is desirable to calculate the amplitudes for these processes from microscopic theory. In this section we give a brief outline of how we derived nonlinear dynamical equations (34) and (35) given in the next section; the details can be found elsewhere [15, 16], including complete definitions of all the notation.

The linear acoustic response collective modes of superfluid \(^3\)He has been found to be well described by a theory which is basically a combination of Landau Fermi liquid theory and BCS theory generalized to p-wave spin-triplet pairing. Calculations have been performed using a number of different formalisms: matrix kinetic equations, path integrals, linear response theory, and quasiclassical Green functions (see ref. [15] for an overview and the original references). We have developed another method, based on the quasiclassical theory of superfluid \(^3\)He [17] which makes the treatment of nonlinear effects tractable. One important aspect of this is the use of Keldysh (nonequilibrium) Green functions which can describe both linear and nonlinear response.

In order to describe the spin degrees of freedom and the particle-hole degrees of freedom of the \(^3\)He quasiparticles, it is convenient to introduce the Nambu 4-spinor:

\[ \Psi = (\psi_\downarrow, \psi_\uparrow, \psi_\uparrow^*, \psi_\downarrow^*) . \] (24)

The Green functions associated with this operator are \(4 \times 4\) matrices. For example, the Matsubara Green function is given by

\[ G^M(x, x', \tau)_{ab} = -\left\langle T_{\tau}(\Psi^\dagger_\tau(x, -\tau)\Psi^\dagger_\tau(x', 0))\right\rangle . \] (25)

The Green functions then have the structure in particle–hole space

\[ \hat{G} = \begin{pmatrix} G & F \\ \overline{F} & \overline{G} \end{pmatrix} , \] (26)

where \(G\) is the conventional one particle Green function which describes the quasiparticles and \(F\) is the anomalous Green function which describes the Cooper pair condensate. The equilibrium
Green function is given by

$$
\hat{G}_0^{\text{NM}}(p, i\epsilon_n) = \frac{-(\xi_n^2 + i\epsilon_n\tau_3 - \Delta)}{\epsilon_n^2 + \xi_n^2 + \Delta(T)^2} \tag{27}
$$

and describes Bogolyubov quasiparticles with energy

$$
E_p = \pm \sqrt{\xi_p^2 + \Delta(T)^2}. \tag{28}
$$

It is convenient to expand the $4 \times 4$ matrices in terms of a basis set \{\gamma_a, a = 1, \ldots, 16\} which generates the SU(4) Lie algebra. This makes the evaluation of traces of products of matrices easier.

The nonequilibrium mean field (self energy) $\delta\sigma$ is written

$$
\delta\sigma(\hat{p}, q, \omega) = \sum_a \gamma_a \delta\sigma_a(\hat{p}, q, \omega). \tag{29}
$$

The diagonal terms (in particle–hole space) describe mean fields associated with density and mass current fluctuations. Fluctuations in the order-parameter are associated with off-diagonal terms. The self energies $\delta\sigma_a$ are related to the nonequilibrium Green functions

$$
\delta\hat{g}(\hat{p}, q, \omega) = \int \frac{d\epsilon}{2\pi i} \int d\xi \delta\hat{G}(p, \epsilon, q, \omega) \tag{30}
= \sum_a \gamma_a \delta\sigma_a(\hat{p}, q, \omega)
$$

by mean field equations of the form

$$
\delta\sigma_a(\hat{p}, q, \omega) = \int \frac{d\Omega'}{4\pi} X_a(\hat{p} \cdot \hat{p'}) \delta\sigma_a(\hat{p}, q, \omega), \tag{31}
$$

where the function $X_a(x)$ is a forward scattering amplitude which can be written in terms of Fermi liquid or pairing parameters for diagonal and off-diagonal terms, respectively. The off-diagonal (in particle–hole space) terms in (31) are time-dependent gap equations which are used to derive the dynamical equations for the order parameter collective modes. The relation between the stress tensor and the collective mode amplitudes (cf. eqs. 5 and 34) is found from the quasiparticle distribution function $\delta\sigma = \frac{1}{4}\text{Tr}(\delta\hat{g})$.

Dyson’s equation expresses the nonequilibrium Green’s function $\delta\hat{G}$ in terms of the equilibrium Green’s function $\hat{G}_0$ and the self energy $\delta\hat{\sigma}$. If this equation is expanded in powers of $\delta\sigma/\Delta$ the result is

$$
\delta\hat{G} = \hat{G}_0 \delta\hat{\sigma} \hat{G}_0 + \hat{G}_0 \delta\hat{\sigma} \delta\hat{G}_0 + \cdots. \tag{32}
$$

The first term describes the linear response and results in eqs. (5) and (6). The second term is response for the nonlinear terms in (34) and (35). The two terms have simple diagrammatic representations (figs. 3 and 4). In fact, it is

Fig. 3. Feynman diagram describing the linear coupling of a zero sound phonon to a squashon (quanta of the $J = 2^-$ mode). Single (double) wavy lines represent phonon (squashon) propagators. The solid lines are propagators for superfluid (Bogolyubov) quasiparticles. Vertices and propagators are all $4 \times 4$ matrices in particle–hole space. The results is proportional to the Tsuneto function defined by eq. (7).
R.H. McKenzie, J.A. Sauls / Nonlinear acoustic effects in superfluid $^3\text{He-B}$

Fig. 4. The nonlinear coupling between two zero sound phonons and a real squashon is calculated in the diagrammatic perturbation theory using this diagram. The result is proportional to the nonlinear coupling strength $A^n$, which appears in eqs. (34) and (35).

4. Excitation of the $J = 2^+$ modes by two sound waves

The propagation of sound, both linear and nonlinear, is described by the wave equation

$$\left(\omega^2 - c_s^2 q^2\right) \delta n(\omega, q) = 2c_s^2 q^2 \delta \Pi(\omega, q),$$

because it follows from the mass and momentum conservation laws. Although this equation is linear in the density fluctuation $\delta n$ and the stress fluctuation $\delta \Pi$, it describes nonlinear sound propagation because $\delta \Pi$ is in general a nonlinear functional of $\delta n$ and the amplitudes of the order parameter collective modes.

We have calculated from the quasiclassical theory the term in $\delta \Pi$ which is second order in $\delta n$ and $\Delta_n$. The result can be written in the form

$$\delta \Pi(\omega) = \frac{1}{(1 + F_\text{ex})} \sum_n \int d\nu A^n(\omega, \nu, \omega - \nu) \times \delta n(\nu) D_n(\omega - \nu),$$

where the nonlinear coupling strength $A^n$ is a dimensionless function of order one and the wave vector dependence has been suppressed for
The dynamical equation for the $J = 2^+$ modes is, to second order in $\delta n$,

$$\lambda(\omega)[\omega^2 + 2i\omega\Gamma - \omega_0(q)^2]D_0(\omega, q) = \frac{6}{N(E_r)^2} \times \int d\nu A'(-\omega, \nu, -\omega) \delta n(\nu) \delta n(\omega - \nu).$$

(35)

A general analytic solution of the wave equation (33) together with the coupled nonlinear equations (34) and (35) is not available. However, considerable insight can be obtained by making use of analysis of similar equations which occur in nonlinear optics [18]. In that case, the wave equation for the electric field $E$ is

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right)E = -4\pi \frac{\partial^2}{\partial t^2} P,$$

(36)

where $P$ is the polarization, and which can be related to the electric field by a nonlinear constitutive relation. Two photon processes are described by equations analogous to (34) and (35) [see, for example, ref. [18], p. 150]. Consequently, Shen’s treatment of two photon absorption and stimulated Raman scattering can be adapted and used here. An important element of this treatment is the slowly varying amplitude approximation. For example the density is written in the form

$$\delta n(R, t) = \text{Re}\{\tilde{N}_j(R, t) + \tilde{N}_j(R, t)\},$$

where

$$\tilde{N}_j(R, t) = N_j(R, t) \exp[i(\omega_j t - q_j \cdot R)],$$

(37)

and the wave amplitudes (or envelope functions) $N_1(R, t)$ and $N_2(R, t)$ vary on time and length scales much longer than the wave period and the wavelength.

If we further assume that the pulse duration is much longer than the mode lifetime, $1/\Gamma$ (i.e. the continuous wave limit) then eq. (35) can be solved for $D_0(R, t)$. It contains terms oscillating with frequencies $0, 2\omega_1, 2\omega_2, \omega_1 + \omega_2$, and $\omega_1 - \omega_2$. The $J = 2^+$ mode will be excited when any of these frequencies is close to the mode frequency. If these solutions for $D_0(R, t)$ together with (37) are substituted in eq. (34) for the stress term $\delta \Pi$, it is found that it contains terms oscillating with frequencies $\omega_1, \omega_2, 3\omega_1, 3\omega_2, 2\omega_1 + \omega_2, 2\omega_2 + \omega_1$. The term with frequency $\omega_1$ is given by

$$\delta \Pi_1(R, t) = [\chi^{(3)}(\omega_1, -\omega_2, \omega_1 + \omega_2) + \chi^{(3)}(\omega_1, \omega_2, \omega_1 - \omega_2)]N_2(R, t)\tilde{N}_1(R, t),$$

where $\chi^{(3)}$ is a nonlinear compressibility given by

$$\chi^{(3)}(\omega, \nu, -\nu) = \frac{6}{N(E_r)^2(1 + F_0)\Delta^2} \left(\frac{c_s q}{\omega}\right)^2 \times \sum_a \lambda(\omega - \nu)(\omega - \nu)^2 + 2i(\omega - \nu)\Gamma - \omega_a(q - s)\right).$$

(38)

This expression together with the wave equation (33) requires that the amplitudes $N_1$ and $N_2$ satisfy the coupled equations

$$\left(\hat{q}_1 \cdot \frac{\partial}{\partial R} + \alpha_1(\omega_1)\right)N_1(R) = -q_1 \kappa(\omega_1, \omega_2)|N_2(R)|^2N_1(R),$$

(39)

$$\left(\hat{q}_2 \cdot \frac{\partial}{\partial R} + \alpha_2(\omega_2)\right)N_2(R) = -q_2 \kappa(\omega_1, \omega_2)|N_1(R)|^2N_2(R),$$

(40)

where

$$\kappa(\omega_1, \omega_2) = \text{Im}[\chi^{(3)}(\omega_1, \omega_2, \omega_1 - \omega_2) + \chi^{(3)}(\omega_1, -\omega_2, \omega_1 + \omega_2)],$$

(41)

and $\alpha_1(\omega)$ is the attenuation of a wave of frequency $\omega$ due to linear coupling to the collective modes (see eq. (9)). If the second sound wave is assumed to be of much higher intensity than the first, then the right hand side of (39) is much smaller than (40) and $N_2(R)$ can be treated as a constant. The first and second waves are then called the signal (s) (or probe) and pump (p) waves, respectively,
and this is known as the undepleted pump wave approximation.

The solution of (39) for constant amplitude 

\[ N_s(R) = N_p \exp(-\alpha_s \cdot R) \]  

(42)

where \( \alpha_s \), the attenuation of the signal wave, is given by

\[ \alpha_s = \alpha_s(\omega_s) + \kappa(\omega_s, \omega_p)|N_p|^2 \]  

(43)

which contains a contribution proportional to the pump wave intensity. To quantify this, we introduce the pump wave energy density, \( U_p \), defined by

\[ U_p = \frac{(1 + F_0)}{N(E_F)} |N_p|^2 \]  

(44)

To leading order in \( (v_F/c_s)^2 \) this can also be written in the same form as the standard expression [19] for the energy density of hydrodynamic sound

\[ U_p = \frac{mc_s^2(N_p)^2}{n} \]  

(45)

The scale to which this will be compared is the superfluid condensation energy density \( U_c \), given by

\[ U_c = \frac{N(E_F) \Delta(T)^2}{2} \]  

(46)

which is typically of the order of \( 10^{-7} \text{ J/cm}^3 \).

There will be features in the attenuation and phase velocity of the signal wave at temperatures \( T_s^n \) and \( T_p^n \) at which the compressibility \( \chi^{(3)} \) has poles; i.e. when

\[ \omega_s \pm \omega_p = \omega_n(T_s^n, q_s, \pm q_p) \]  

(47)

The plus and minus signs correspond to two phonon absorption and nonlinear Raman effects, respectively.

The nonlinear coupling function \( A^n \) can be factored into the form

\[ A^n(\omega, \nu, \omega - \nu) = \frac{1}{2} Z^n(q_s, q_p) \tilde{A}(\omega, \nu, \omega - \nu, T) \]  

(48)

The frequency and temperature dependence is contained in the function \( \tilde{A} \), which can be expressed in terms of Tsuneto functions (see eq. (7)) \( \lambda(\omega), \lambda(\nu), \) and \( \lambda(\omega - \nu) \). The complete expression is given in eqs. (153), (154), and (162) in ref. [16]. The function \( \tilde{A} \) is dimensionless and typically in the range 0.1–0.5. The dependence on the tensor structure of the collective mode with eigentensor \( u_{\mu \nu}^s \) and the direction of the two sound waves is contained in the factor \( Z^n \) given by

\[ Z^n(q_s, q_p) = (\hat{q} \cdot \hat{s}) u_{\mu \nu}^s \hat{s} \hat{q}_\mu - \frac{1}{2} u_{\mu \nu}^s \hat{s} \hat{q}_\mu - \frac{1}{2} u_{\mu \nu}^s \hat{q}_\mu \hat{q}_\mu \]  

(49)

When \( \omega_s + \omega_p = \omega_n \), i.e. for resonant two phonon absorption, there is a peak with magnitude

\[ \Delta \sigma_n = \frac{3\omega_s}{50(\omega_s + \omega_p)} \frac{\Delta^2(T_s)|Z^n|^2}{c_0 \Gamma(1 + F_0)^2} U_p \]  

(50)

where

\[ \Delta = |\tilde{A}(\omega_s, -\omega_p, \omega_s + \omega_p, T)|^{1/2}\lambda(\omega_s + \omega_p, T) \]  

(51)

It is important to note that the Fermi liquid parameter \( F_0 \) increases from about 10 at 0 bar to 100 at 34 bar. Consequently, these nonlinear effects will be largest at low pressures.

Note that the nonlinear features are inversely proportional to the mode damping \( \Gamma \). As discussed in section 3, this becomes smallest at low temperatures. Hence, the nonlinear features will be largest when the sound frequencies are chosen so that the three wave resonance occurs at low temperatures.

*There are two misprints in eq. (153) in ref. [16]. The definitions of \( A \) and \( E \) should read \( A(\omega, \nu, \omega - \nu) = -4 T k_s \cdot (\omega - \nu) k_s; E(\omega, \nu, \omega - \nu) = -(2/\pi \eta) \omega(\omega - \nu) \Delta(\omega - \nu) \).
4.1. Excitation of the $J = 2^+$ modes by two parallel sound waves

In this case, eq. (49) for the coupling strength $Z_n$ gives

\[ Z_n = \frac{1}{2} u_\mu^\nu \hat{q}_\mu \hat{q}_\nu, \]

which gives the same angular dependence as the expression (14) for the linear excitation of the $J = 2^+$ modes. Hence, in zero field only the $m = 0$ mode will be excited. The coupling strength is then

\[ Z_0^1 = \sqrt{\frac{2}{27}}. \]

The peak for two-phonon absorption occurs at a temperature $T_0$ such that

\[ (\omega_0 + \omega_p)^2 = a \Delta(T_0) + v_0^2(q_s + q_p)^2, \]

where $v_0$ is the $m = 0$ mode velocity. As the magnetic field is increased, nonlinear evolution of the mode frequencies as a function of field will be observed. At fields much larger than the crossover field $H_0$, defined by eq. (15), a linear Zeeman splitting will be observed. For $H < \frac{1}{2} H_0$ the $M = \pm 2(-2)$ mode will have the highest (lowest) frequency and the largest (smallest) attenuation peak.

4.2. Excitation of the $J = 2^+$ modes by two perpendicular sound waves in zero field

The $J = 2^+$ modes are excited with wave vector $q_s + q_p$, which defines the mode quantization axis in the nonlinear case. The coupling strength given by equation (49) is

\[ Z_m^\perp = -\frac{1}{2} (u_\mu^\nu \hat{q}_\mu \hat{q}_\nu + u_\mu^\nu \hat{q}_\mu \hat{q}_\nu). \]

Explicit calculation gives

\[ Z_0^\perp = 1/\sqrt{54}, \quad Z_{\pm 1}^\perp = 0, \quad Z_{\pm 2}^\perp = \frac{1}{6}. \]

Hence, only the $m = 0$ and $m = \pm 2$ modes are excited. It is important to note that the coupling strength (somewhat surprisingly) does not depend on the relative magnitude of $q_s$ and $q_p$, which determines the direction of $q_s + q_p$. There will be peaks at temperatures $T_0^\perp$ and $T_\pm^\perp$, due to two phonon absorption by the $m = 0$ and $m = \pm 2$ modes respectively, given by

\[ (\omega_0 + \omega_p)^2 = a \Delta(T_0^\perp)^2 + v_0^2(q_s^2 + q_p^2), \]

\[ (\omega_0 + \omega_p)^2 = a \Delta(T_\pm^\perp)^2 + v_0^2(q_s^2 + q_p^2). \]

Note that comparing eqs. (54) and (57) which determine the temperatures $T_0^\perp$ and $T_\pm^\perp$, it follows that their separation is related to the mode velocity $v_0$ and provides a means to determine it. In one-phonon processes, order parameter modes can only be resonantly excited at a single wave number $q$ given by

\[ (c_0 q)^2 = a \Delta(T)^2 + v_0^2 q^2. \]

(i.e. where the sound and mode dispersion relations cross; cf. fig. 1). Hence, it is not possible to determine the mode dispersion in linear experiments. However, for two-phonon processes, the magnitude of the wave vector of excitation can be varied by changing the relative directions of the two sound waves. This provides a means to determine the mode dispersion. A similar approach has been used in nonlinear optical spectroscopy. Two-photon absorption has been used to measure the dispersion of excitons in semiconductors [18, p. 206]. If the heights of the attenuation peaks at $T_\pm$, $T_0^\perp$, and $T_\pm^\perp$ are denoted by $\Delta \alpha_\perp$, $\Delta \alpha_0^\perp$, and $\Delta \alpha_\pm^\perp$, respectively, then it follows from (53) and (56) that they are related by

\[ \Delta \alpha_\parallel = 4 \Delta \alpha_0^\perp = \frac{1}{4} \Delta \alpha_\pm^\perp. \]

Hence the sum of all the peak heights is the same for parallel and perpendicular waves. It should be pointed out that in an actual experiment the sound pulses are of finite duration and so the overlap (interaction time) is different for parallel and perpendicular pulses and must be taken into account.

In order to calculate the magnitude of attenuation of the signal wave due to two phonon
absorption by the $J = 2^+$ mode, an expression for the mode lifetime $1/\Gamma$ is required. Wölfle [20] calculated the damping of the modes due to quasiparticle collisions in terms of the quasiparticle scattering amplitudes. However, the mode lifetime is always of the same order of magnitude as the quasiparticle lifetime. It is known [21] that at very low temperatures ($T \ll T_c$) the temperature dependence of the lifetime is given by

$$\frac{1}{\tau} = A(P) \left( \frac{T_c}{T} \right)^{3/2} \exp \left( -\frac{\Delta(T)}{T} \right),$$

where the coefficient $A$ is a function of pressure.

In our calculations [16, 22] we have assumed a simple form for the temperature dependence of the mode lifetime

$$\Gamma = \Gamma_0(T_c/T)^{3/2} \exp \left( -\frac{\Delta(T)}{T} \right),$$

where $\Gamma_0$ depends on pressure. This is not meant to be a quantitative model of the mode lifetime, but contains the correct qualitative behavior. Hence, we stress that the height and width of the peaks in figs. 2 and 5 (where we took $\Gamma_0/kT_c = 0.007$) should not be used for quantitative comparison with experiment. On the other hand, the oscillator strength or area under the peaks is independent of the mode lifetime and can be compared quantitatively with experiment.

The predicted temperature dependence of the attenuation of a 36 MHz signal wave in the presence of a parallel 3.2 MHz pump wave with energy density $0.2 U_c$ is shown in fig. 5. This can be compared to fig. 2 which has the same parameters, but no pump wave is present. The large central peak at $T_R$ is due to linear coupling between the $J = 2^+$ mode and signal wave. The nonlinear features at $T_+$ and $T_-$ are due to two-phonon absorption and the inverse Raman effect, respectively. The background attenuation is due to linear coupling to the $J = 2^-$ mode off-resonance.

5. Observation of two-phonon absorption by the $J = 2^+$ modes

Nonlinear acoustic experiments are difficult to perform, in part because of the heating of the superfluid caused by the high sound intensities required. The temperature width of the collective mode resonances are typically of the order of a microkelvin. Hence, the cryostat must be sufficiently good to prevent temperature drifts of this magnitude during the propagation of the sound pulses.

A group in Helsinki has recently observed two-phonon absorption by the $J = 2^+$ modes in two different sound cells [23–25]. We briefly describe the essential details of these experiments. A more complete discussion can be found in refs. [23] and [24]. The first sound cell is the same one used in previous studies of collective modes in rotating superfluid $^3$He [26]. The cell is a cylinder of radius 3 mm and height 4 mm. The sound propagates along the cylinder axis between two quartz crystal transducers with fun-
fundamental frequency of 8.895 MHz. A magnetic field could be applied parallel to the cylinder axis. Parallel pump and signal wave pulses of 12 µs duration were used. The sound energy density in the pump wave was determined calorimetrically: the temperature increase caused by a large number of pulses as a function of transducer voltage was measured.

The observed temperature dependence of the attenuation of a 26.8 MHz signal wave in the presence of a parallel 8.9 MHz pump wave is shown in fig. 6 in a narrow temperature range near \( T_c = 0.72 T_c \) for several pump wave energy densities. Note that there are actually two peaks, not one. A smaller peak occurs at a temperature 13 ± 3 µK below the large peak. The theory predicts that in the absence of a magnetic field only the \( J = 2^+, m = 0 \) mode should be excited by two parallel sound waves. The height of both peaks is linearly proportional to the pump wave energy density, as predicted by eq. (5) (see fig. 7).

Table 1 shows the temperature \( T_+ \) at which two-phonon absorption has been observed for a

![Fig. 6](image)

Fig. 6. Observation of two phonon absorption by the \( J = 2^+ \) mode [23]. The temperature dependence of the attenuation of a 26.8 MHz signal wave in the presence of a parallel 8.9 MHz pump wave is shown for three different pump wave energy densities. The temperature \( T_+ = (0.72 T_c) \) at which the large peak occurs is close to the predicted value. The smaller peak is due to two phonon absorption involving the reflected wave (see text).

![Fig. 7](image)

Fig. 7. The observed height of the new attenuation peak is proportional to the pump wave energy density. The frequencies of the signal and pump waves are 26.8 and 8.9 MHz, respectively. The solid line is a line of best fit. Its slope is a measure of the strength of the nonlinear coupling between the \( J = 2^+ \) modes and two sound waves.

<table>
<thead>
<tr>
<th>Data set</th>
<th>Pressure (bar)</th>
<th>( f_s ) (MHz)</th>
<th>( f_p ) (MHz)</th>
<th>( T_+/T_c )</th>
<th>( h(f_s + f_p)/\omega_0(T_c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sound cell 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>26.8</td>
<td>8.9</td>
<td>0.72 ± 0.02</td>
<td>1.00 ± 0.02</td>
</tr>
<tr>
<td>2</td>
<td>2.7</td>
<td>44.7</td>
<td>8.9</td>
<td>0.56 ± 0.03</td>
<td>0.98 ± 0.02</td>
</tr>
<tr>
<td>3</td>
<td>3.5</td>
<td>44.7</td>
<td>8.9</td>
<td>0.64 ± 0.02</td>
<td>0.97 ± 0.02</td>
</tr>
<tr>
<td><strong>Sound cell 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>25.15</td>
<td>15.1</td>
<td>0.53 ± 0.02</td>
<td>0.96 ± 0.01</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>15.1</td>
<td>5.0</td>
<td>0.93 ± 0.02</td>
<td>1.0 ± 0.1</td>
</tr>
</tbody>
</table>

Table 1: The temperature \( T_+ \) at which two phonon absorption was observed to occur for parallel pump and signal waves [23, 24]. The last column gives the ratio of the sum of the two sound wave frequencies to the \( J = 2^+, m = 0 \) mode frequency at \( T_+ \), \( \omega_0(T_+) \) predicted by eq. (61). The corresponding quantity for linear response experiments has been found to have values in the range 0.95–0.98 (see fig. 16, in ref. [1]). The Greywall temperature scale [30] has been used.
number of different pressures and pump and signal wave frequencies. The temperature $T_+$ should be given by eq. (54). If we use the fact that $q = \omega / c_1$, then to leading order in $(v_F / c_1)^2$, neglecting Fermi liquid corrections,

$$\omega_s + \omega_p = \omega_0(T) = \sqrt{8/5} \Delta(T) \left( 1 + \frac{1}{2} \left( \frac{v_0}{c_1} \right)^2 \right).$$

(61)

At low pressures the second term is of the order of 2% of the first term. The last column in table 1 contains values for this ratio $(\omega_0(T_+)) / \omega_0(T)$. To test whether it is actually two-phonon absorption that is being observed, the values should be compared to previous measurements of the $J = 2^+$, $m = 0$ mode frequency by linear ultrasound experiments. The ratio $\omega / \omega_0(T)$ has been found to be weakly temperature dependent. At low pressures, and for $T \lesssim 0.7 T_c$, it has been found to have values in the range 0.95–0.98 [1, fig. 16].

We now consider the explanation of the small peak seen in fig. 6. It could be due to another $J = 2^+$ mode with $m \neq 0$. A small dispersion splitting of these modes is expected (cf. eq. (17)). However, application of a magnetic field should increase the splitting of the peaks and change their relative heights. It was found that a field of up to 8 mT did not change significantly either the height or position of the peaks. Another alternative is that it is due to a superflow splitting of the mode. A recently observed splitting of the $J = 2^-$ mode was attributed to a superflow resulting from heat currents in the sound cell [27]. However, this can be ruled out because the splitting was unchanged by rotating the superfluid at speeds up to 1 rad/s.

An alternative explanation for the small peak has been suggested independently by Volovik and by Shen. The front of the pump wave sound pulses will be reflected off the receiving transducer and interact with the end of the signal wave pulse. Consequently, the $J = 2^+$, $m = 0$ mode will be excited at wave vector $q_s - q_p$ at frequency $\omega_s + \omega_p$ in addition to excitation at $q_s + q_p$. If this is the case, the relative height of the two peaks depends on the overlap between the incident signal wave and the reflected pump wave when the pulse duration is increased the overlap increases and so does the ratio of the height of the small peak to that of the large peak. The temperature $T_+$ at which this will occur is given by

$$(\omega_0 + \omega_p)^2 = a\Delta(T_+)^2 + v^2_S(q_s - q_p)^2.$$  

(62)

As a result of the small dispersion of the $J = 2^+$ mode the temperature $T_+$ will be slightly lower than the temperature $T_+$ at which two-phonon absorption occurs for the incident wave. Equations (54) and (62) can be combined to give the collective mode velocity $v_0$ in terms of the temperature splitting $(T_+ - T_+)$

$$v_0 = [k B (T_+ - T_+) \frac{\sqrt{\Delta T}}{T_c} \left( \frac{1}{f_s} \right) + \frac{1}{f_p}]^{1/2},$$

(63)

where $\Delta(T/T_c)$ is the magnitude of the slope of the curve $\Delta(T/T_c)/kT_c$ versus $T/T_c$. We have used the approximation $q_s = \omega_0 / c_0$ and $q_p = \omega_p / c_0$.

The splitting was observed to be $T_+ - T_+ = 13 \pm 3 \mu K$. For $a = 8/5$ the result is $v_0 / c_0 = 0.20 \pm 0.03$. This will be compared to the theoretical predictions for the mode velocity elsewhere [25].

Figure 7 shows a graph of the observed height of the signal-wave attenuation peak due to two-phonon absorption as a function of the pump wave energy density for dataset 1. Clearly the relationship is linear, as predicted by eq. (50). The slope $S$ of this line is a measure of the nonlinear coupling between the $J = 2^+$ mode and the two sound waves. For parallel sound waves, $|Z_s|^2 = (2/27)\delta_{\omega_0}$ and only the $m = 0$ mode is excited. According to (50) the slope $S$ is given by

$$S = \left( \frac{f_s \Delta(T_+)}{225(1 + F_0^2)\hbar f_s f_p c_1 k \sqrt{\Delta T} \left( \frac{T_+}{T_c} \right) \delta T} \right),$$

(64)

where $\delta T$ is the linewidth in temperature given by (11). The nonlinear coupling $\%$ can be calcu-
lated for the different temperatures $T_+$ and sound frequencies using the expressions given in ref. [16]. The slope of the line in fig. 7 is $3.4 \pm 0.3 \text{cm}^{-1}$. From fig. 6 the line width is $2.3 \pm 0.5 \mu \text{K}$, giving $S \delta T = 8 \pm 3 \text{cm}^{-1} \mu \text{K}$. For the parameter values for dataset 1, the nonlinear coupling function $\xi = 0.12$ and the theoretical prediction for $S \delta T$ is $24 \text{cm}^{-1} \mu \text{K}$. Hence, the observed value is three times smaller than the theoretical value. A possible source of this discrepancy is the finite duration of the sound pulse. The $2 \mu \text{K}$ line width for dataset 1 corresponds to a mode lifetime of about 10 ns. Since this is comparable to the pulse duration (12 ns) the continuous-wave limit does not hold. Hence, as discussed in section 2, the observed attenuation will be smaller than for the continuous wave limit. A complete comparison of the predicted coupling strengths with the observed results will be given elsewhere [25]. We stress that this requires accurate measurements of the line width.

Further experiments have now been performed in a second sound cell [24, 25]. It is a cube with sides $9 \text{mm}$ long. Two pairs of faces of the cube are quartz transducers. Hence it is possible to propagate sound in two perpendicular directions. Both transducers can be excited at odd multiples of their fundamental frequency of 5.0 MHz. A magnetic field can be applied parallel to the direction of propagation of one transducer pair. Hence, with parallel pump and signal waves it should be possible to observe the five-fold splitting of the $J = 2^+$ mode in a transverse magnetic field. Two-phonon absorption has been observed in this cell under a number of different conditions. The different parameter values are summarized in table 1. Two-phonon absorption by the $J = 2^+$ modes has been observed in zero field with the directions of the pump and signal waves parallel and perpendicular to one another [25].

The results are as expected: for perpendicular waves, two peaks corresponding to the $m = 0$ and $m = \pm 2$ modes are seen at temperatures $T^0_\perp$ and $T^{\perp}_\perp$, respectively. The mode velocity $v_0$ can be found from the temperature splitting, $T_+ - T^0_\perp$ between the location of the $m = 0$ mode for parallel and perpendicular waves. Combining eqs: (54) and (57) gives (for $a = 8/5$ and $q_s = \omega_s / c_0$, $q_p = \omega_p / c_0$)

$$
\frac{v_0}{c_0} = \left[ \frac{\sqrt{8/5}}{\hbar} \frac{k}{\hbar} (T_+ - T^0_\perp) \Delta' \left( \frac{T_+}{T_c} \right) \left( \frac{1}{f_s} + \frac{1}{f_p} \right) \right]^{1/2}.
$$

(65)

The splitting of the $m = 0$ and $m = \pm 2$ modes can be used to determine the mode velocity $c_b$:

$$
\frac{c_b}{c_0} = \left[ 3 \frac{\sqrt{8/5}}{\hbar} \frac{k}{\hbar} (T^0_\perp - T^{\perp}_\perp) \Delta' \left( \frac{T^0_\perp}{T_c} \right) \left( \frac{f_s + f_p}{f_s^2 + f_p^2} \right) \right]^{1/2}.
$$

(66)

This can be compared to the determination of $c_b$ by Shivaram et al. [6] from the textual splitting (compare eq. (21)). A complete comparison between the measured mode velocities and the theoretical values will be given elsewhere [25].

Two-phonon absorption was also observed for parallel pump and signal waves in transverse magnetic fields up to $40 \text{mT}$. The expected five-fold Zeeman splitting of the peak has been seen [24]. This is further confirmation that it is the $J = 2^+$ modes that are excited by the two sound waves. As noted in section 2, at low pressures, one is in the dispersion dominated regime for magnetic fields up to $50 \text{mT}$. Hence, there is an asymmetry in the coupling strengths of the different modes. In particular, the $M = +2$ mode has the largest coupling strength, as expected.

Attempts have been made to observe the Raman effects predicted at temperature $T_+$ given by $\omega_s - \omega_p \sim \sqrt{8/5} \delta(T_+)$. Unfortunately, for all the possible combinations of frequencies of the sound transducers, $T_+$ is close to the temperature $T_i$ at which the higher frequency wave has a linear resonance with the $J = 2^-$ mode (i.e. $\omega_s$ or $\omega_p \sim \sqrt{12/5} \delta(T_i)$). The fact that this mode couples so strongly to sound prevents observation of the much smaller nonlinear effects in this temperature range.

Two phonon absorption by the $J = 2^-$ mode should be extremely small due to the near exact particle–hole symmetry of the $^3\text{He}$-Fermi liquid (see section 2). Searches for this effect have yielded no results, as would be expected.
6. Future opportunities

The observation of two-phonon absorption by the \( J = 2^+ \) modes shows the feasibility of nonlinear acoustic experiments in \(^3\text{He-B}\). There are many other acoustic analogues of nonlinear optical effects which should be investigated.

We have discussed previously \([16, 22]\) stimulated Raman scattering by the \( J = 2^+ \) mode. This will occur when \( \omega_p - \omega_s \approx \sqrt{8/5} \Delta(T) \) (\( \omega_p > \omega_s \)). The signal wave will be amplified because the higher frequency pump phonons decay into real squashons and phonons with the frequency of the signal wave. We have also discussed anti-Stokes waves (\( 2\omega_s - \omega_p \)) and third-harmonic generation (\( 3\omega_s \)) and how their intensity will be reduced due to dispersion. The third-harmonic generation will be largest when the pump wave frequency \( \omega_p \) is such that \( 2\omega_p \approx \sqrt{8/5} \Delta(T) \). However, this has the undesirable implication that \( \sqrt{12/5} \Delta(T) < 3\omega_p < 2\Delta \). It can be seen from fig. 2 that in this temperature range, the linear attenuation is of order \( 10 \text{ cm}^{-1} \). (Although the wave is not damped by pair breaking, it is still strongly damped by the \( J = 2^+ \) mode, off resonance.) The third-harmonic signal will only be measurable if the pump wave is of sufficient intensity that the gain due to nonlinear processes is larger than the attenuation of the third-harmonic wave due to linear processes. Since the latter is so large, it may not be possible to generate a detectable signal.

In ref. \([15]\), four-wave mixing was discussed. Two antiparallel pump waves propagate perpendicular to two antiparallel signal waves. This configuration can be used to produce phase-conjugated waves and backward parametric amplification and oscillation. Four-wave mixing of light has been used to generate squeezed states of light. In ref. \([28]\) we discussed the possibility of observing squeezed phonon states in \(^3\text{He-B}\).

Two-phonon absorption by the \( J = 2^+ \) modes occurs because the pump wave breaks the particle–hole symmetry of the equilibrium ground state. In a similar manner, in the presence of a superflow, there will be a coupling between the \( J = 2^+ \) modes and sound. This effect has not yet been observed but is worthy of further investigation. For further discussion, see ref. \([29]\).

An effect we have not discussed previously is the propagation of a single high intensity sound wave. If the wave is propagating in the \( z \) direction, the spatial dependence of its amplitude \( N(z) \) is given by the differential equation

\[
\left( \frac{\partial}{\partial z} + \alpha_L(\omega) \right) N(z) = -q\kappa(\omega, \omega) N(z)^3,
\]

which is obtained by setting \( N_1 = N_2 = N(z) \) in eq. (39). If \( 2\omega \approx \sqrt{8/5} \Delta(T) \) then two-phonon absorption by the \( J = 2^+ \) mode will occur, making \( \kappa \) large. In this frequency region \( \alpha_L(\omega) \) will be small. We assume that the wave is of sufficient intensity that \( \kappa N^2 > \alpha_L \). The solution of (67) is then given by

\[
N(z) = \frac{1}{N(0)^2 + 2q\kappa(\omega, \omega)z}.
\]

It is useful to define the transmission \( T(\omega) \) through a path length \( L \),

\[
T(\omega) = \left( \frac{N(L)}{N(0)} \right)^2.
\]

There will be a minimum in the transmission when \( 2\omega = \omega_s(T) \). It follows from (68) that the minimum transmission \( T_{\text{min}} \) is related to the sound wave energy density \( U \) by

\[
\frac{1}{T_{\text{min}}} = 1 + m \frac{U}{U_c},
\]

where \( m \), the slope of a graph of \( 1/T_{\text{min}} \) versus \( U/U_c \), is given by

\[
m = \frac{\tilde{A}(\omega_s - \omega)^2 \Delta(T)^2}{225\lambda(2\omega)c_0 T(1 + F_0)^3} L.
\]

Note that \( m/L \) will be of the same order of magnitude of the slope of the line in fig. 7. This experiment will be more difficult to perform than the two wave experiments because the heating will be greater since a high intensity wave is being attenuated.
7. Conclusions

Collective modes of the superfluid order parameter cause sharp resonances in the acoustic response. We have shown that the approximate particle–hole symmetry of the $^3\text{He}$-Fermi liquid determines selection rules for linear and nonlinear coupling of sound to order-parameter collective modes.

We have formulated the quasiclassical theory of superfluid $^3\text{He}$ so that the calculation of the nonlinear coupling between zero sound and the collective modes is tractable. It involves a diagrammatic perturbation theory which exploits the symmetries of $^3\text{He}$-Fermi liquid. The end result is nonlinear dynamical equations that are similar to those found in nonlinear optics. Consequently, we predicted measurable two-phonon absorption and stimulated Raman scattering by the $J=2^+$ modes at low pressures.

Pekola et al. [23–25] have recently seen distinct features in the nonlinear acoustic response in two different sound cells. These features can be identified with two-phonon absorption by the $J=2^+$ modes because (a) the peaks in the signal wave attenuation occur at the predicted temperature, (b) the peak height is proportional to the pump wave energy density, and (c) in a magnetic field there is a five-fold splitting of the peaks.

Two-phonon absorption can be used to determine the dispersion of the $J=2^+$, $m=0$ mode, something that is not possible in linear acoustic experiments. This is an example of one of the exciting new developments we look forward to because $^3\text{He}$-B is such an ideal medium in which to observe acoustic analogues of nonlinear optical effects.

Acknowledgements

We are particularly grateful to Jukka Pekola, Kiyoshi Torizuka, Antti Manninen, Jukka Kynääräinen, and Harry Alles for providing us with their experimental results prior to publication. We have benefited from many discussions with them. We have also had useful discussions with Bill Halperin and John Wilkins. Research at The Ohio State University was supported by a University-Battelle Memorial Fellowship and the DOE (Grant no. DE-FGO2-88-ER45347). Research at Northwestern University was supported by the NSF (Grant no. DMR-88-09854) through the Science and Technology Center for Superconductivity.

References


