Anisotropy of the Upper Critical Field in a Heavy Fermion Superconductor

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We study the upper critical field for various possible order parameters of a heavy fermion superconductor. We pay a particular attention to the question how the anisotropy of upper critical field changes by effects of paramagnetism, impurity scattering, and fermi surface anisotropy in the presence of strong spin-orbit coupling. The paramagnetic limit can have a dramatic effect on the anisotropy in odd-parity superconductors, and provide important information on the symmetry of underlying order parameters. We also discuss the effect of impurity scattering.

I. INTRODUCTION

The upper critical field, $H_{c2}$, of heavy fermion superconductors shows many novel features. A single crystal $UPt_3$ has the unusual temperature-dependence at low temperature, as well as a kink-structure near the transition temperature. For $U_{1-x}Th_xBe_{13}$ the upper critical field rises steeply in the Ginzburg-Landau region and deviates quite early from its linear slope. We have previously presented a theoretical explanation for the unusual temperature-dependence of $H_{c2}$ for $UPt_3$ and discussed how the paramagnetic effect can provide crucial information on the spin-structure of underlying order parameters in heavy fermion superconductors (referred to as CS). In this paper we present detailed calculations of the upper critical field for heavy fermion superconductors whose crystal symmetry belong to the hexagonal point group $D_{6h}$, which include $UPt_3$, $UNi_2Al_3$ and $UPd_2Al_3$. We compute the upper critical field for all temperature, while taking into account key elements to determine the anisotropy of $H_{c2}$ in heavy fermion superconductors; anisotropy of order parameters, fermi surface anisotropy, paramagnetic effect, impurity scattering as well as the strong spin-orbit coupling.

The anisotropy of $H_{c2}$ of pure superconductor is determined by the anisotropy of fermi surface near the transition temperature, where the paramagnetic coupling is unimportant. At lower temperatures, however, the paramagnetic limit can have a dramatic effect on the anisotropy of $H_{c2}$ in odd-parity superconductors. For even-parity states, the upper critical field is bounded by the paramagnetic effect for all directions of field. But for odd-parity states, there is no suppression of superconductivity if the external field is along with the direction of Cooper-pair spin; whereas magnetic field of other orientations will be a pair-breaking source. We also study the effect of impurity scattering on $H_{c2}$. Nonmagnetic impurities can be pair-breaking even in the absence of applied field for the unconventional superconductors whose order parameters belong to the non-identity representations. This leads to change the slope of $H_{c2}$ near the transition temperature and the anisotropy of $H_{c2}$.

In Section II we derive general equations for the upper critical field, while incorporating an arbitrary fermi surface, paramagnetic effect, fermi-liquid corrections, impurity scattering, and the unconventional order parameters. In Section III we discuss the effect of paramagnetism on the anisotropy of $H_{c2}$ in clean limit. The effect of impurity scattering is considered in Section IV.

II. FORMALISM

We use the quasiclassical theory of superconductivity and follow closely the notation of CS and Alexander, et al. The key quantity to compute is the quasiclassical propagator $\hat{g}(k_f, \mathbf{R}, \epsilon)$, which is a $4 \times 4$ matrix in the Nambu representation:

$$\hat{g}(k_f, \mathbf{R}, \epsilon) = \left( \begin{array}{cc} g + \frac{g \cdot \sigma}{i\sigma_y} [f + f \cdot \sigma]i\sigma_y & \left[ f + f \cdot \sigma \right] \right),$$ (1)

where $\bar{g} = g^*(-k_f, \mathbf{R}, \epsilon)$, $\bar{g} = g^*(-k_f, \mathbf{R}, \epsilon)$, $\bar{f} = f^*(-k_f, \mathbf{R}, \epsilon)$, $\bar{f} = f^*(-k_f, \mathbf{R}, \epsilon)$, and $\sigma = (\sigma_z, \sigma_y, \sigma_z)$, are the Pauli matrices in spin space. Here $\epsilon$ is the Matsubara frequency, $\mathbf{R}$ is the real space position, and $k_f$ is a two dimensional coordinate defining the position on the fermi surface. The quasiclassical propagator satisfies a transport-like equation:

$$[i\epsilon \hat{\tau}_3 - \hat{\Sigma}(k_f, \mathbf{R}, \epsilon) - \hat{v}_{ext}(k_f, \mathbf{R}) \cdot \hat{\tau}_3] \hat{g}(k_f, \mathbf{R}, \epsilon) + i\mathbf{v}_f(k_f) \cdot \nabla_R \hat{g}(k_f, \mathbf{R}, \epsilon) = 0,$$ (2)

$$\hat{g}^2 = -\pi^2 \mathbf{1}.$$ (3)

The fermi velocity $v_f(k_f)$ depends on $k_f$ and $\hat{\tau}_3$ is a Pauli matrix in particle-hole space. The coupling to a magnetic field is given by:

$$\hat{v}_{ext}(k_f, \mathbf{R}) = \frac{c}{e} \mathbf{v}_f(k_f) \cdot \mathbf{A}(\mathbf{R}) \cdot \hat{\tau}_3 + \mu(k_f) \mathbf{S} \cdot \mathbf{B}(\mathbf{R}).$$ (4)

The first term is the orbital coupling to the field, while the second term is the paramagnetic coupling to the spin. Here $e$ and $\mu(k_f)$ are electric charge and effective magnetic moment of a quasiparticle, $c$ is speed of light.
light. The vector potential $\mathbf{A}(\mathbf{R})$ generates the total field $\mathbf{B} = \nabla \times \mathbf{A}$, and $\hat{S}$ is the spin operator:

$$\hat{S} = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_y \sigma_y \end{pmatrix}. \quad (5)$$

The self-energy $\hat{\Sigma}$ includes the pairing self-energy, $\Delta(k_f, \mathbf{R})$, as well as the self-energies for impurity scattering and fermi-liquid corrections. One simplifying feature is that we need the propagator $\hat{g}$ only to the first order in $\Delta$ in order to compute the upper critical field, so that the diagonal component of $\hat{g}$ can be replaced by the normal state propagator. The fermi-liquid corrections can then be absorbed in the definition of effective magnetic moment $\mu(k_f)$. The equation for impurity self-energy can be written

$$\hat{\Sigma}_{imp}(k_f, \mathbf{R}, \epsilon) = \langle w(k_f, k'_f) \hat{g}(k'_f, \mathbf{R}, \epsilon) \rangle_{k'_f}, \quad (6)$$

where the bracket $\langle \cdots \rangle_{k_f}$ denotes a fermi surface integration over the variable $k_f$, and $w(k_f, k'_f)$ is a scattering probability of nonmagnetic impurities. Note that eq.(6) is valid beyond the Born approximation in our calculation.\textsuperscript{12}

By solving the off-diagonal component of $\hat{g}$ from eqs. (2-3) together with the self-consistent equation for impurity self-energy and the weak-coupling gap equation, we obtain the following coupled equations: for odd-parity states,

$$\Delta(k_f, \mathbf{R}) = T \sum_{\epsilon} \langle V^{odd}(k_f, k'_f) f(k'_f, \mathbf{R}, \epsilon) \rangle_{k'_f}, \quad (7)$$

where

$$f(k_f, \mathbf{R}, \epsilon) = 2\pi \int_0^\infty d\tau \exp(-\tau \hat{L}) \times \left\{ \begin{array}{l}
\left[ 1 + \{\cos(2\tau \mu H) - 1\} \hat{h} \hat{h}' \right] \times \\
\left[ \Delta(k_f, \mathbf{R}) + \langle w(k_f, k'_f) f(k'_f, \mathbf{R}, \epsilon) \rangle_{k'_f} \right]
\end{array} \right\}, \quad (8)$$

$$f(k_f, \mathbf{R}, \epsilon) = 2\pi \int_0^\infty d\tau \exp(-\tau \hat{L}) \times \left\{ \begin{array}{l}
\cos(2\tau \mu H) \langle w(k_f, k'_f) f(k'_f, \mathbf{R}, \epsilon) \rangle_{k'_f} \\
-\text{sgn}(\epsilon) \sin(2\tau \mu H) \times \\
\hat{h}' \cdot \left[ \Delta(k_f, \mathbf{R}) + \langle w(k_f, k'_f) f(k'_f, \mathbf{R}, \epsilon) \rangle_{k'_f} \right]
\end{array} \right\}. \quad (9)$$

The operator $\hat{L}$ is defined by

$$\hat{L} = 2 \mid \epsilon \mid + 2\pi w_0 + \text{sgn}(\epsilon) \mathbf{v}_f \cdot \mathbf{\hat{\partial}}, \quad (10)$$

where $\mathbf{\hat{\partial}} = \nabla_R + i(2e/c)A$ and $w_0$ is the $s$-wave part of $w(k_f, k'_f)$. The pairing interaction is denoted by $V^{odd}(k_f, k'_f)$, the direction of magnetic field by the unit vector $\hat{h}$, and its transpose by $\hat{h}' = (h_x, h_y, h_z)$. Note that singlet and triplet components of the propagator, $f$ and $\hat{f}$, decouple without the paramagnetic terms. The equations for even-parity states can be obtained from eqs. (7-9) with the following substitutions:

$$f \leftrightarrow \hat{f}, \quad \Delta \leftrightarrow \Delta, \quad \hat{h} \leftrightarrow \hat{h}', \quad V^{odd} \leftrightarrow V^{even}. \quad (11)$$

The upper critical field is then computed as the largest value of $H$ for which eq.(7) has a nontrivial solution.

For odd-parity, the scalar order parameter satisfies the same equation as in eq. (7-9) with the following substitutions:

$$\Delta(k_f, \mathbf{R}) = 2\pi T \sum_{\epsilon} \left\{ \begin{array}{l}
\lim_{\epsilon \to 0} \exp \left\{ -2\tau \mid \epsilon \mid -\text{sgn}(\epsilon) \tau \mathbf{v}_f \cdot \mathbf{\hat{\partial}} \right\}
\end{array} \right\} \times \left\{ \begin{array}{l}
\left[ 1 + (\cos(2\tau \mu H) - 1) \hat{h} \hat{h}' \right] \cdot \Delta(k_f, \mathbf{R}) \right\}_{k'_f}. \quad (12)$$

For even-parity, the scalar order parameter satisfies the similar equation to eq. (12) with a substitution, $\hat{h} \hat{h}' \to 1$. Note that the paramagnetic term is unimportant near the transition temperature for both odd- and even-parity states; however, it can have a large effect on $H_{c2}$ at low temperatures, except for the case of odd-parity states with $\Delta \perp \mathbf{H}$.

For numerical calculations we use the same material parameters as in CS with an extension to take into account impurity scattering: (i) two fermi velocities, $v_f^\parallel$ and $v_f^\perp$, to parametrize our uniaxial model for the fermi surface, (ii) an isotropic effective magnetic moment, $\mu$,\textsuperscript{13} (iii) an isotropic part of impurity scattering, $w_0$, (iv) superconducting transition temperatures, $T_c$ and $T_{co}$, with and without impurities (in zero field), respectively. We also introduce the coherence lengths, $\xi_{\perp} = v_f^\perp/2\pi T_{co}$, $\xi_{\|} = v_f^\parallel/2\pi T_{co}$, and a magnetic scale, $H_o = \hbar c/(2e)\xi_{\perp}^2$, as well as the dimensionless parameters:

$$\eta = \left( \frac{\xi_{\|}}{\xi_{\perp}} \right)^2, \quad \bar{\mu} = \frac{\mu H_o}{\pi T_{co}}, \quad \bar{w}_0 = \frac{w_0}{T_{co}}, \quad \bar{t}_c = \frac{T_c}{T_{co}}. \quad (13)$$

The upper critical field is calculated with $\mathbf{H}$ along $\hat{z}$, the axis of six fold symmetry, and also for $\mathbf{H}$ in the basal plane.

### III. PARAMAGNETIC EFFECT

We compute the upper critical field in the clean limit. First we consider the odd-parity state: the 2D representations, $E_{1u}$ and $E_{2u}$, and also the $B_{1u}$ representation,
which for our purpose is representative of all the 1D odd-parity states.\textsuperscript{7} For the $E_{1u}$ state the pairing potential and order parameter are:

$$V^\text{odd}(k_f, k_f) = \lambda \hat{z} \hat{z}^* \left[ \psi^*(k_f) \psi(k_f) + \psi(k_f) \psi^*(k_f) \right],$$

(14)

$$\Delta(k_f, R) = \hat{z} \left[ \eta_+(R) \psi^*(k_f) + \eta_-(R) \psi(k_f) \right],$$

(15)

where $\lambda$ is a coupling constant and $\psi(k_f) \sim k_x + i k_y$. Note that $\Delta$ is along $\hat{z}$, so that the Cooper pair spins lie in the basal plane. By putting eqs. (14-15) into the gap equation (12), we obtain equations for the amplitudes $\eta_{\pm}(R)$:

$$\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} = \begin{pmatrix} K_{on} & K_{off} \\ K_{off} & K_{on} \end{pmatrix} \begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix},$$

(16)

where

$$K_{on} = 4 \pi T \int_{\epsilon \geq 0}^\infty \int_0^\infty d\tau \left| \psi \right|^2 \exp(-2 \tau \epsilon - \tau \mathbf{v}_f \cdot \partial \mathbf{r}) \times [1 + \{ \cos(2 \tau \mu H) - 1 \} \cos^2(\theta_h)].$$

(17)

The angle between magnetic field $\mathbf{h}$ and $\hat{z}$ axis of the crystal is denoted by $\theta_h$. The matrix element $K_{off}$ has the same expression as $K_{on}$ except that $\left| \psi \right|^2$ is replaced by $\psi^2$. We solve the equation (16) for all temperature by a standard method of introducing raising and lowering operators, $\hat{a}_\pm$, and a set of eigenfunctions, $\{ \phi_n(R) \}$, of the harmonic oscillator problem.\textsuperscript{14}

When the magnetic field is along $\hat{z}$, we define $\hat{a}_+ = \frac{1}{2}(\partial_x + i \partial_y)$ and $\hat{a}_- = \frac{1}{2}(\partial_x - i \partial_y)$, and then expand $\eta_{\pm}(R)$ in terms of $\{ \phi_n(R) \}$:

$$\eta_{\pm}(R) = \sum_{n=0}^\infty P_{\pm n} \phi_n(R).$$

(18)

Putting eq. (18) into eq. (16) yields a matrix equation for $P_{\pm n}$. This can be diagonalized in a block form and its eigenvalues are obtained from the equations:

$$\alpha_0 = 1,$$

(19)

$$(1 - \alpha_n)(1 - \alpha_{n+2}) = \beta_n^2, \quad \text{for } n \geq 0,$$

(20)

where

$$\alpha_n = 4 \pi T \int_{\epsilon \geq 0}^\infty \int_0^\infty d\tau \left| \psi \right|^2 \exp(-2 \tau \epsilon - \frac{1}{2} p^2) \times \cos(2 \tau \mu H) L^0_n \left( \frac{p^2}{\sqrt{(n+1)(n+2)}} \right),$$

(21)

$$\beta_n = 4 \pi T \int_{\epsilon \geq 0}^\infty \int_0^\infty d\tau \left| \psi \right|^2 \exp(-2 \tau \epsilon - \frac{1}{2} p^2) \times \cos(2 \tau \mu H) \frac{p^2 L^0_n \left( \frac{p^2}{\sqrt{(n+1)(n+2)}} \right)}{\sqrt{(n+1)(n+2)}},$$

(22)

with $p^2 = e H(\tau v_f)^2/c$, and $L^0_n$ is a generalized Laguerre polynomial. Here we have used the formula:

$$\exp(-\tau \mathbf{v}_f \cdot \partial) \phi_n(R) = \sum_{m=0}^\infty \sum_{l=0}^\infty \exp\left\{ -\frac{p^2}{2} + i \theta(m - l) \right\} \times \frac{p^{m+l}(-1)^m \sqrt{n!(n-m+l)!}}{m!(n-m)!} \phi_{n-m+l}(R),$$

(23)

where $\theta$ is the azimuthal angle of the fermi velocity, $\mathbf{v}_f$, in basal plane. The maximum value of $H_{c2}$ occurs for $(\eta_+, \eta_-) \sim (\phi_2, c_0 \phi_0)$ with $\alpha = \beta_0/(1 - \alpha_0)$, that is,$^{15}$

$$\Delta(k_f, R) \sim \hat{z}[\phi_2(R)(k_x - i k_y) + c_0 \phi_0(R)(k_x + i k_y)],$$

(24)

The limiting values of $H_{c2}$ at $T = 0$ and close to $T_{c0}$ are listed in the table I of CS.

When the magnetic field lies in the basal plane, the upper critical field is independent of paramagnetism because the Cooper pairs have no amplitude with zero spin projection for directions in the plane. As a consequence of a uniaxial symmetry of the fermi surface $H_{c2}$ is isotropic in the basal plane at all temperatures, and we can choose a direction of magnetic field along the $x$ axis. It is then convenient to work in a basis of $(\eta_1, \eta_2)$ in which equations for $\eta_1$ and $\eta_2$ decouple. Here $\eta_1 = (\eta_+ + \eta_-)/\sqrt{2}$ and $\eta_2 = -i(\eta_+ - \eta_-)/\sqrt{2}$. The equation for $\eta_1$ becomes

$$\frac{1}{\lambda} \eta_1(R) = 4 \pi T \int_{\epsilon \geq 0}^\infty \int_0^\infty d\tau \left( \psi_1(k_f)^2 \exp(-2 \tau \epsilon - \tau \mathbf{v}_f \cdot \partial) \right) \eta_1(R),$$

(25)

where $\psi_1 \sim k_x$. A similar equation holds for $\eta_2$ with $\psi_2 \sim k_y$. We solve the eq. (25) by a variational method, i.e. by introducing a parameter $\alpha$ in the operators of harmonic oscillator,

$$\hat{a}_\pm = \pm \frac{1}{2} (\partial_x \mp i \alpha \partial_y),$$

and their corresponding eigenfunctions $\{ \phi_{n}(R, \alpha) \}$.

The maximum $H_{c2}$ occurs when $\eta_1 = \phi_0(R, \alpha)$ with $\alpha = 1/\sqrt{\eta}$, which corresponds to a rescaling of the fermi surface and exact diagonalization of Eq. (25). As a remark we note that the eigenvalue equation for $\eta_2$ can not be solved exactly by a simple rescaling of fermi surface because the exact eigenfunction is a linear combination of all $\phi_{2n}$’s with $n \geq 0$. However it turns out that a simple trial function, $\eta_2 = \phi_0(R, \alpha)$ with an optimum value of $\alpha$, provides a good estimate for $H_{c2}$. Addition of other terms such as $\phi_2$ and $\phi_4$ in the trial function does not increase the value of $H_{c2}$ by more than one percent.
a trial function of the form \( \eta \) in units of \( H \), and has a maximum eigenvalue for \( \mu = 0 \). Note that \( H_{c2}^{\parallel} \) is suppressed by the paramagnetic coupling at low temperature, while \( H_{c2}^{\perp} \) is independent of \( \mu \). We can easily make \( H_{c2}^{\parallel} \) and \( H_{c2}^{\perp} \) cross over with a suitable choice of parameters, as shown in the bottom two curves.

In fig. 1 we summarize the results for \( E_{1u} \) representation for several choices of the effective mass ratio \( \eta \) and scaled effective moment \( \tilde{\mu} \). Both \( h_{c2}^{\parallel} \) and \( h_{c2}^{\perp} \) are scaled in units of \( H_0 \); in which \( h_{c2}^{\parallel} \) is independent of \( \eta \), while \( h_{c2}^{\perp} \) scales as \( 1/\tilde{\eta} \). As noted earlier the paramagnetic effect can significantly reduce the value of \( h_{c2}^{\parallel} \) at low temperatures, whereas it has no effect on \( h_{c2}^{\perp} \). Thus, by adjusting the values of \( \eta \) and \( \tilde{\mu} \), we can easily make the upper critical fields, \( h_{c2}^{\parallel} \) and \( h_{c2}^{\perp} \), cross over each other at finite temperature and fit the experimental data quite well.

For the \( E_{2u} \) representation the order parameter,

\[
\Delta(k_f, R) = \frac{\tilde{x} + i\tilde{y}}{\sqrt{2}} \eta_+(R) \psi(k_f) - \frac{\tilde{x} - i\tilde{y}}{\sqrt{2}} \eta_-(R) \psi^*(k_f),
\]

with \( \psi(k_f) \sim k_x + i k_y \), differs significantly in its spin structure from that of \( E_{1u} \). Close to \( T_{c0} \) we have

\[
h_{c2} = \frac{5}{14\zeta(3)} (\cos^2 \theta_h + \frac{\eta}{2} \sin^2 \theta_h)^{-\frac{3}{2}} (1 - t),
\]

for an arbitrary direction of the field. Here \( \zeta(3) \) is the Riemann zeta function. When \( H \parallel \tilde{z} \), following the similar steps as in the \( E_{1u} \) representation, we find that the upper critical field is independent of the paramagnetic term and has a maximum eigenvalue for \( (\eta_+, \eta_-) \sim (\tilde{\phi}_0, 0) \). For fields in the basal plane, \( H_{c2} \) is sensitive to \( \tilde{\mu} \) and we use a trial function of the form \( (\eta_1, \eta_2) \sim (\tilde{\phi}_0, 0) \) for a variational calculation. We summarize the results in fig. 2.

We can obtain a weak crossover of the upper critical fields even without the paramagnetic term for a limited range of \( \eta \), \( 2 < \eta < 3 \); however, an important point is that \( H_{c2} \) becomes more isotropic as \( \tilde{\mu} \) increases.

There are four odd-parity, 1-D representations in the limit of strong spin-orbit coupling, all of which have a similar spin structure to that of \( E_{1u} \) representation, and therefore exhibit similar anisotropic paramagnetic effects. For the \( B_{1u} \) representation \( \Delta(k_f, R) = \tilde{z} \eta(R) \psi(k_f) \) with \( \psi \sim k_x^3 - 3k_x k_y^2 \). Close to \( T_{c0} \) we obtain

\[
h_{c2} = \frac{9}{14\zeta(3)} (\cos^2 \theta_h + \frac{\eta}{4} \sin^2 \theta_h)^{-\frac{3}{2}} (1 - t),
\]

Away from \( T_{c0} \) we use a trial function \( \tilde{\phi}_0 \) to calculate \( H_{c2} \) for both principal directions of the field; \( H_{c2} \) depends on \( \tilde{\mu} \) only for \( H \parallel \tilde{z} \). A numerical calculation shows that we can make the anisotropy ratio of upper critical fields, \( h_{c2}^{\parallel}/h_{c2}^{\perp} \), almost identical to that of \( E_{1u} \) by a suitable choice of parameters, \( \eta \) and \( \tilde{\mu} \).

The most important distinction of even-parity states is that the paramagnetic term is important for any direction of the magnetic field. For the \( E_{2g} \) representation,

\[
\Delta(k_f, R) = \eta_+(R) \psi(k_f) + \eta_-(R) \psi^*(k_f),
\]

where \( \psi(k_f) \sim (k_x + i k_y)^2 \). When \( H \parallel \tilde{z} \) the upper critical field can be computed exactly with \( (\eta_+, \eta_-) \sim (\tilde{\phi}_4, \tilde{c}_0 \tilde{\phi}_0) \) with a constant \( \tilde{c}_0 \). For \( H \parallel \tilde{x} \) we perform a variational calculation with \( \eta_1 = \tilde{\phi}_0(R, \alpha) \). As shown in fig. 3, a very weak crossover is possible for a small
range of parameter, e.g. $3.0 < \eta < 4.2$ for $\bar{\mu} = 0$. This is so small that its weak crossover may be an artifact of the variational calculation. We find that the inclusion of paramagnetic term further reduces its anisotropy. Any anisotropy that would exist in the absence of the paramagnetic coupling is reduced because the paramagnetic limit has a greater effect for the direction in which the upper critical field is larger. Similar results are obtained for other even-parity representations.

It can be shown that $d(\epsilon)$ and $e(\epsilon)$ are a real-valued odd and even function of $\epsilon$, respectively.

By putting eqs. (31-33) into eqs. (7-9) and using the formula (23), we obtain a set of coupled equations for $a$, $b$, $d$, and $e$:

$$\frac{a}{\lambda} = 4\pi T \sum_{\epsilon \geq 0} \int_0^\infty d\tau \langle \exp \{-2\tau(\epsilon + \pi \omega_0) - \frac{p^2}{2}\} \rangle \left[ \cos(2\tau \mu H) \{a \langle 1 - 2p^2 + \frac{p^2}{2}\rangle + b \langle p^2 / \sqrt{2}\rangle \} \langle \psi \rangle^2 \right. $$

$$+ \omega_0 \langle \sqrt{2p} - \frac{p^3}{\sqrt{2}} \rangle \langle \psi | Re \{I(\epsilon) \exp(2i\tau \mu H)\} \rangle \right] \right], \quad (33)$$

$$\frac{b}{\lambda} = 4\pi T \sum_{\epsilon \geq 0} \int_0^\infty d\tau \langle \exp \{-2\tau(\epsilon + \pi \omega_0) - \frac{p^2}{2}\} \rangle \left[ \cos(2\tau \mu H) \{a \langle p^2 / \sqrt{2}\rangle + b \} \langle \psi \rangle^2 \right. $$

$$- \omega_0 \langle p \rangle \langle \psi | Re \{I(\epsilon) \exp(2i\tau \mu H)\} \rangle \right] \right], \quad (34)$$

where $Re \{ \}$ denotes a real part of the argument and $I(\epsilon) = d(\epsilon) + ie(\epsilon)$. The equation for $I(\epsilon)$ becomes:

$$I = 2\pi \int_0^\infty d\tau \langle \exp \{-2\tau(\epsilon + \pi \omega_0) + 2i\tau \mu H - \frac{p^2}{2}\} \rangle \left[ \{a(-\sqrt{2p} + \frac{p^3}{\sqrt{2}}) + bp \} \langle \psi \rangle + \omega_0 (1 - p^2) I \right] \rangle. \quad (35)$$

The upper critical field is the largest eigenvalue of $H$ in eqs. (34-36). The results are summarized in fig. 4. Note that the impurity scattering reduces the values of $H_{c2}$ from the clean limit at all temperatures. Paramagnetic coupling limits the upper critical fields away from $T_c$ and reduces range of the Ginzburg-Landau region where $H_{c2}$ depends linearly on temperature. Close to $T_c$ we obtain

$$\frac{H_{c2}^\parallel}{T_c} = \frac{5(3 + \sqrt{5})}{21\zeta(3)} \left( \frac{31\pi^4}{1008\zeta(3)} \frac{\bar{\omega}_0}{T_c} \right) \left( 1 - \frac{t}{t_c} \right), \quad \bar{\omega}_0 \ll 1. \quad (36)$$
For $\mathbf{H} \parallel \hat{c}$ the upper critical field is independent of the paramagnetism and the impurity self-energy terms vanish, $\langle \mathbf{f} \rangle = \langle \mathbf{f} \rangle = 0$. The upper critical field is then computed by a simple substitution of $\epsilon \rightarrow \epsilon + \pi \omega_0$ in the gap equation of the pure case, eq. (12). Close to $T_c$ we derive a general expression:

$$\frac{dh_{c2}}{dt} = t_c \sum_{n \geq 0} \frac{S(0)}{S(\omega_0/T_c)} \left[ \frac{dh_{c2}^{\parallel}}{dt} \right]_{\text{pure}},$$  \hspace{1cm} (37)

where

$$S(x) = \sum_{n \geq 0} (2n + 1 + x)^{-3}.$$  \hspace{1cm} (38)

Note that eq. (38) is valid for any cases in which $\langle \mathbf{f} \rangle = \langle \mathbf{f} \rangle = 0$. This includes the non-identity representations of even-parity states such that $\langle \Delta(k_f) \rangle = 0$. In the plot of $H_{c2}/H_0$ vs. $T/T_c$, the slope of $H_{c2}$ at $T_c$ is generally smaller than that of pure superconductor due to a large reduction of $T_c$ by impurity scattering. On the contrary the impurity scattering can enhance the values of $H_{c2}$ for the identity representation in which there is no reduction of $T_c$. In fig. 5 we show the effect of impurity scattering on the anisotropy of $H_{c2}$. The values of $H_{c2}$ are reduced by impurity scattering for both directions, while the crossover point moves towards lower temperature. This might explain the fact that some experiments for $UPt_3$ do not show a distinct crossover at finite temperature.\textsuperscript{2,3}

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A numerical calculation shows that anisotropy of the magnetic susceptibility does not lead to a large change in the anisotropy of $H_{c2}$. For example, an anisotropy of 50% in the susceptibility, $\chi_\parallel / \chi_\perp = 0.5$, yields less than a 10% change of $H_{c2}^\parallel / H_{c2}^\perp$ for a s-wave superconductor at $T=0$.

